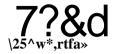
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QUADRATIC VOLUME PRESERVING MAPS: AN EXTENSION OF A RESULT OF MOSER

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A natural generalization of the Hénon map of the plane is a quadratic diffeomorphism that has a quadratic inverse. We study the case when these maps are volume preserving, which generalizes the the family of symplectic quadratic maps studied by Moser. In this paper we obtain a characterization of tliese maps for dimension four and less. In addition, we use Moser's result to construct a subfamily of in n dimensions.

1. Introduction

Some of the simplest nonlinear systems are given by quadratic maps: for example the logistic map in one dimension and the quadratic map introduced by Hénon [14, 15] in the plane. It is easy to see that any quadratic, one dimensional map with a fixed point is affinely conjugate to the logistic map, $xy^{-n} rx(-x)$. In a similar way, Hénon showed that a generic quadratic area-preserving mapping of the plane can be written in normal form as

$$k + y + x^{\wedge}$$
) - (-x

2

which has a single parameter k.

Hénon's study can be generalized in several directions. Moser [22] studied a class of quadratic symplectic maps, having obtained a useful decomposition and normal form. For example, when the map is quadratic and symplectic in M[^], Moser [22,19] showed that it can be written as the composition of twon in 5549 in o49 Tw0.127 Tc(dimensiona) 4(a) Tj1

where W is a homogeneous cubic polynomial in p. The map given in (1) is a particular example of what we call a quadratic shear.

Definition 1. A quadratic shear is a bijective map of the form

$$X^{\wedge} fix) = X + -Qix), \tag{2}$$

where Q(x) is a vector of homogeneous, quadratic polynomials such that $f^{-\wedge}$ is also a quadratic map.

In this way Moser's result is basically a characterization of all symplectic quadratic shears. One of the remarkable aspects of this is that quadratic symplectic maps necessarily have quadratic inverses. In general we can write a quadratic map on E" as the composition of an affine map with a quadratic map that is zero at the origin and is the identity at linear order:

$$x \stackrel{\wedge \Psi}{=} fix) = xo + L\{x + Q\{x\}),$$
 (3)

where SQ S M", L is a matrix, and Qix) is a vector of homogeneous, quadratic polynomials. Note that if the map / is volume preserving then it is necessary that L satisfies det(L) = 1. Similarly if / is symplectic, then L must be a symplectic matrix. Of course, the quadratic terms also can not be chosen arbitrarily in these cases.

Polynomial maps are of interest from a mathematical perspective. Much work has been done on the "Cremona maps", that is polynomial maps with constant Jacobians [8]. An interesting mathematical problem concerning such maps maps",

ii)=>i) By assumption, det(D/(a;)) and det(D/ $^{(/(a;))}$) are polynomials in xi,X2,-.- ,a:,,. However, differentiation of $f_{-}^{(f(x))} = x$ gives

$$det{Dr}(x))detiDf{x})=^l,$$

and therefore, since both are polynomials, $det{Df(x)}$ has to be a constant independent of x. We notice that $det{Df(x)} = det{D/(0)} = det(/) = 1$.

i)=>iii) Since det($(/ + M\{x)) = 1$ and M is linear in x, then for any C 7^{\lapha} 0

$$det\{M\{x\}-CI\} = (-1)^{"}C^{"}det(/ + M(-^{a};)) = (-1)^{"}C^{"}.$$

•

This implies that the characteristic polynomial of M(x) is (-C)" and therefore [M(a;)]'' = 0.

At this point, we restrict to the case of quadratic maps in standard form

We will see that for the

A map / is symplectic with respect to w if u {Dfv, Dfv') = u{v,v'} for all

4. Dimensions Three and Four

Following Coroliary 1, we would like to establish the stronger result that $M(a;)^{\wedge} = 0$ for all x. In this section x.

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