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K.E.LENZ
Department of Mathematics and Statistics
University of Minnesota
Duluth, MN 55812

H. E. LOMELÎ
Depeurtment of Mathematics
Instituto Tecnològico Autônomo de Mexico
Mexico, DF 01000

J.D.MEISS
Department of Applied Mathematics
University of Colorado
Boulder, CO 80309
El-mail: jdm®boulder.colorado.edu

QUADRATIC VOLUME PRESERVING MAPS: AN EXTENSION OF A RESULT OF MOSER

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A natural generalization of the Hénon map of the plane is a quadratic diffeomorphism that has a quadratic inverse. We study the case when these maps are volume preserving, which generalizes the the family of symplectic quadratic maps studied by Moser. In this paper we obtain a characterization of tliese maps for dimension four and less. In addition, we use Moser's result to construct a subfamily of in n dimensions.

1. Introduction

Some of the simplest nonlinear systems are given by quadratic maps: for example the logistic map in one dimension and the quadratic map introduced by Hénon [14, 15] in the plane. It is easy to see that any quadratic, one dimensional map with a fixed point is affinely conjugate to the logistic map, *xy-^ rx{\ — x).* In a similar way, Hénon showed that a generic quadratic area-preserving mapping of the plane can be written in normal form as

$$:) - (\begin{matrix} k + y + x^{\wedge} \\ -x \end{matrix}$$

which has a single parameter k.

Hénon's study can be generalized in several directions. Moser [22] studied a class of quadratic symplectic maps, having obtained a useful decomposition and normal form. For example, when the map is quadratic and symplectic in M^", Moser [22,19] showed that it can be written as the composition of twon in 5549 in o49 Tw0.127 Tc(dimensiona) 4(a) Tj1 where W is a homogeneous cubic polynomial in *p*. The map given in (1) is a particular example of what we call a quadratic shear.

Definition 1. A quadratic shear is a bijective map of the form

X ^ fix) = X + *-Qix),* (2)

where *Q{x)* is a vector of homogeneous, quadratic polynomials such that *f~^* is also a quadratic map.

In this way Moser's result is basically a characterization of all symplectic quadratic shears. One of the remarkable aspects of this is that quadratic symplectic maps necessarily have quadratic inverses. In general we can write a quadratic map on E" as the composition of an affine map with a quadratic map that is zero at the origin and is the identity at linear order:

x ^¥ fix) = xo + L{x +-Q{x)), (3)

where SQ S M", L is a matrix, and *Qix)* is a vector of homogeneous, quadratic polynomials. Note that if the map / is volume preserving then it is necessary that *L* satisfies det(L) = 1. Similarly if / is symplectic, then *L* must be a symplectic matrix. Of course, the quadratic terms also can not be chosen arbitrarily in these cases.

Polynomial maps are of interest from a mathematical perspective. Much work has been done on the "Cremona maps", that is polynomial maps with constant Jacobians [8]. An interesting mathematical problem concerning such maps maps",

ii)⇒i) By assumption, $\det(Df(x))$ and $\det(Df^{-1}(f(x)))$ are polynomials in $x_1, x_2, \ldots, x_n$. However, differentiation of $f^{-1}(f(x)) = x$ gives

$$\det(Df^{-1}(f(x)))\det(Df(x)) = 1,$$

and therefore, since both are polynomials, $\det(Df(x))$ has to be a constant independent of x. We notice that $\det(Df(x)) = \det(Df(0)) = \det(I) = 1$.

i)⇒iii) Since $\det(I + M(x)) = 1$ and M is linear in x, then for any $C \neq 0$

$$\det(M(x) - CI) = (-1)^n C^n \det(I + M(-x/C)) = (-1)^n C^n.$$

This implies that the characteristic polynomial of $M(x)$ is $(-C)^n$ and therefore $[M(x)]^n = 0$. ∎

At this point, we restrict to the case of quadratic maps in standard form

We will see that for the

A map / is symplectic with respect to w if $u\}\{Dfv,Dfv')$ = $u\{v,v')$ for all

4. Dimensions Three and Four

Following Coroliary 1, we would like to establish the stronger result that M(a;)^ = 0 for all *x*. In this section x.

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K. E. JIEH^, H. E. JIOMEJIH, *RTK.* fl. MEftC

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