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Some Topology of the 3-Body Problem

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The purpose of this paper is to topologically characterize the surfaces of constant momentum, angular momentum, and energy occuring in the planar 3-body problem. For most values of these parameters the integral [redacted]

1. FORMULATION OF THE PROBLEM

Three particles of mass m_1, m_2, and m_3, respectively, move in the plane under the influence of their mutual gravitational attraction. Let $q_1 = (x_1, x_2)$, $q_2 = (x_3, x_4)$ and $q_3 = (x_5, x_6)$ denote the positions of the particles and let [redacted] Let $p_1 = (y_1, y_2)$, $p_2 = (y_3, y_4)$, and $p_3 = (y_5, y_6)$ denote the momenta of the three particles. Thus, the state of the system is specified by a point $(x, y) \in R^6 \times R^6$. Let $K = \{x \in R^6 : r_{ij}(x) = 0 \text{ for some } ij\}$, and let the gravitational constant $G = 1$. Then the equations of motion can be formulated as a Hamiltonian system with Hamiltonian $H : (R^6 - K) \times R^6 \to R^1$ defined by

[redacted]

where

$$U(x) = \sum_{i>j} m_i m_j r_{ij}^{-1}(x).$$

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The equations of motion are

[redacted]

The well-known integrals of these equations are the integrals of linear momentum, angular momentum and energy and our goal is to describe the surfaces of constant momentum, angular momentum and energy. Without loss of generality we will assume that the linear momentum is zero ($\sum_{K=1}^{3} p_K = 0$) and that the center of mass of the system is located at the [redacted] putation that $m_1 = m_2 = m_3 = 1$, although the techniques we develop here also apply to the general case.

[redacted]

The integral of angular momentum $J : R^6 \times R^6 \to R^1$ is defined by

$$J(x, y) = (x_1 y_2 - x_2 y_1) + (x_3 y_4 - x_4 y_3) + (x_5 y_6 - x_6 y_5).$$

The surface of constant angular momentum ω (and zero linear momentum) is denoted by $M^7(\omega)$. Let

$$A_1 = (1, 0, 1, 0, 1, 0),$$
$$A_2 = (0, 1, 0, 1, 0, 1),$$

[redacted]

Define

$$M^7(\omega) = \{(x, y) \in R^6 \times R^6 : |x| \neq 0,\ A_1 \cdot x = A_2 \cdot x = 0,$$
$$\text{and } A_3(x) \cdot y = \omega,\ A_1 \cdot y = A_2 \cdot y = 0\}.$$

PROPOSITION 2.1. *$M^7(\omega)$ is diffeomorphic to $S^3 \times R^4$.*

Proof. Define $\rho : M^7(\omega) \to S^5$ by $\rho(x, y) = x\,|x|^{-1}$, and let $\hat{S}^3 = \rho(M^7(\omega)) = \{u \in S^5 : A_1 \cdot u = A_2 \cdot u = 0\}$. Clearly, $\hat{S}^3$ is diffeomorphic to S^3. If $u \in \hat{S}^3$, then $\rho^{-1}(u) = \{(x, y) : x\,|x|^{-1} = u,\ |x|\,A(u)\,y = \Omega\}$ where $\Omega = \mathrm{col}(0, 0, \omega)$. The matrix $A(u)$ has rank 3 for each $u \in \hat{S}^3$ and therefore $\{y : |x|\,A(u)y = \Omega\}$ is a 3-plane in R^6. Hence $\rho^{-1}(u)$ is diffeomorphic to $R^1 \times R^3$. The map ρ is locally trivial and thus $M^7(\omega)$ is a smooth 4-plane fibre bundle over S^3. However, every 4-plane bundle over S^3 is trivial and hence $M^7(\omega)$ is diffeomorphic to $S^3 \times R^4$.

3. SURFACES OF CONSTANT ENERGY

The total energy of the three bodies is represented by the Hamiltonian function H. Define

$$M^6(h, \omega) = \{(x, y) \in M^7(\omega) : H(x, y) = h\}.$$

[redacted]
$\pi(M^6(h, \omega))$.

LEMMA 3.1. *Let $u \in \hat{S}^3$. Then $u \in M^3(h, \omega)$ if and only if $U^2(u) + 2h\omega^2 \geqslant 0$.*

[redacted]

$$|\, y \,|^2 = 2h + 2U(x).$$

Since $(x, y) \in M^7(\omega)$ we must have $A(x)y = \Omega$. We also have $|\, y\,|^2 \geqslant \omega^2 \,|\, x \,|^{-2}$ since the minimum of $|\, y\,|^2$ on the set $Y = \{v : A(x)\, v = \Omega\}$ is $\omega^2 \,|\, x\,|^{-2}$. Since $U(x) = |\, x\,|^{-1}\, U(u)$, we obtain the inequality

$$2h \,|\, x\,|^2 + 2U(u)\,|\, x\,| - \omega^2 \geqslant 0.$$

In order for this inequality to hold we must have $U^2(u) + 2h\omega^2 \geqslant 0$.

On the other hand, if $U^2(u) + 2h\omega^2 \geqslant 0$, then there exists $\lambda > 0$ such that

$$2h\lambda^2 + 2U(u)\,\lambda - \omega^2 \geqslant 0.$$

For $x = \lambda u$, the set of y satisfying the equations $A(x)\, y = \Omega$ and $|\, y\,|^2 = 2h + 2U(x)$ is nonempty. Thus, if $x = \lambda u$ and if y satisfies these equations, then $(x, y) \in M^6(h, \omega)$ and $\pi(x, y) = u$. This completes the proof.

Let $S^2 = \{s = (s_1, s_2, s_3) : s_1^2 + s_2^2 + s_3^2 = 1\}$ and define $P : \hat{S}^3 \to S^2$ by

$$P(u) = 1/\sqrt{3}(r_{12}(u), r_{13}(u), r_{23}(u)).$$

P is well defined since for each $u \in \hat{S}^3$, $r_{12}^2(u) + r_{13}^2(u) + r_{23}^2(u) = 3$.

[redacted]

(1) $s_1 > 0,\ s_2 > 0,\ s_3 > 0$,

(2) $s_1 + s_2 \geqslant s_3\,,\ s_1 + s_3 \geqslant s_2\,,\ s_2 + s_3 \geqslant s_1\,,$

(3) $V^2(s) + 2h\omega^2 \geqslant 0$.

The set of $s \in S^2$ satisfying conditions (1) and (2) above is a triangle ∇ on S^2 minus its corners. The level lines V = constant appear on this triangle as pictured in Fig. 1 below.

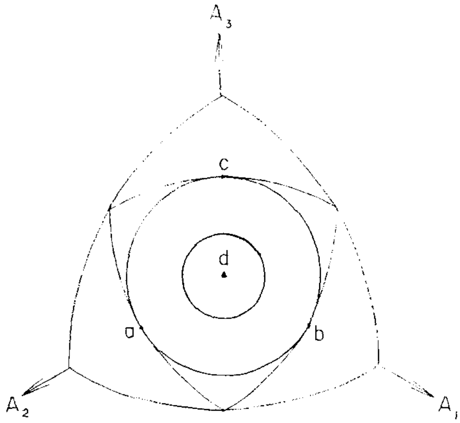

FIGURE 1

Here $d = 3^{-1/2}(1, 1, 1)$, $a = 6^{-1/2}(1, 2, 1)$, $b = 6^{-1/2}(2, 1, 1)$, and [redacted]

(a) $-2h\omega^2 < V^2(d)$;

[redacted]

PROPOSITION 3.4. *$M^3(h, \omega)$ is diffeomorphic to $S^3 - 3S^1$ if inequality A holds, to $S^3 - 3S^1 - 2(T - \partial T)$ if inequality B holds, and to $(T - S^1) \cup (T - S^1) \cup (T - S^1)$ if inequality C holds.*

Proof. Suppose inequality B is satisfied. Then $M^2(h, \omega)$ is equal to the triangle ∇ minus its corners and minus an open disk containing the point d. Thus $M^3(h, \omega) = P^{-1}(M^2(h, \omega))$ is equal to S^3 minus the union of three S^1's and two solid open tori. It is possible to verify directly that these circles and tori are pairwise linked and unknotted. The other cases are treated similarly.

We are now ready to classify the integral surfaces $M^6(h, \omega)$ for various values of h and ω.

[redacted]

$$\alpha(x, y) = (x \mid x \mid^{-1}, \mid x \mid, y \mid y \mid^{-1}).$$

α is clearly an imbedding and $\alpha(M^6(h, 0)) = p^{-1}[(\hat{S}^3 - 3S^1) \times R^+]$. Hence $M^6(h, 0)$ is diffeomorphic to $(S^3 - 3S^1) \times R^1 \times S^2$.

[redacted]

$\beta(x, y) = (x \mid x \mid^{-1}, y)$. β is an imbedding and $\beta(M^6(h, 0)) = (P')^{-1}[\hat{S}^3 - 3S^1]$. Hence, for $h < 0$, $M^6(h, 0)$ is diffeomorphic to $(S^3 - 3S^1) \times R^3$. This completes the proof.

PROPOSITION 3.6. *Suppose that $-2h\omega^2 < V(d)$ and suppose $\omega \neq 0$, $h \geqslant 0$. Then $M^6(h, \omega)$ is diffeomorphic to $(S^3 - 3S^1) \times R^3$.*

Proof. Let $E = \{(u, v) \in \hat{S}^3 \times R^6 : A(u)v = \Omega\}$, where $\Omega = \text{col}(0, 0, \omega)$. E is the total space of a 3-plane fibre bundle over $\hat{S}^3$ and hence is diffeomorphic to $S^3 \times R^3$. For $h \geqslant 0$, define $\alpha : M^6(h, \omega) \to E$ by $\alpha(x, y) = (x \mid x \mid^{-1}, y \mid x \mid)$. α is an imbedding since $\alpha^{-1}(u, v) = (\mid x \mid u, \mid x \mid^{-1} v)$, where $\mid x \mid$ is the unique positive solution to the equation $\mid v \mid^2 = [2h + 2 \mid x \mid^{-1} U(u)] \mid x \mid^2$. It is easy to check that the image of α is diffeomorphic to $(S^3 - 3S^1) \times R^3$ (the inequality $-2h\omega^2 < V(d)$ insures that $M^3(h, \omega) = (S^3 - 3S^1)$) and hence $M^6(h, \omega)$ is diffeomorphic to $(S^3 - 3S^1) \times R^3$.

Proposition 3.7. *Suppose that* $\omega \neq 0$ *and suppose* $M^3(h, \omega)$ *is diffeomorphic to* $(S^3 - 3S^1)$ *(inequality A holds). If* $h < 0$ *then* $M^6(h, \omega)$ *is diffeomorphic to* $(S^3 - 3S^1) \times$ p**Ŝ**e.

Proof. Let $u \in M^3(h, \omega)$. We want to show that $\pi^{-1}(u)$ is a 3-sphere. If $(x, y) \in \pi^{-1}(u)$, then $x = \lambda u$ for some scalar λ and the equations $\lambda A(u) y = \Omega$ [redacted] plane with smallest norm. Therefore λ must satisfy the inequality $|y(\lambda)|^2 = \omega^2\lambda^{-2} \leqslant |y|^2 = 2h + 2\lambda^{-1}U(u)$. Thus, for each $u \in M^3(h, \omega)$ the admissible λ's must lie in the interval $[a(u), b(u)]$, where

$$a(u) = [(U^2(u) + 2h\omega^2)^{1/2} - U(u)]\, h^{-1},$$

$$b(u) = -[(U^2(u) + 2h\omega^2)^{1/2} + U(u)]\, h^{-1}.$$

For each $\lambda \in (a(u), b(u))$, the set of $y \in R^6$ satisfying the equations $\lambda A(u) y = \Omega$, and $|y|^2 = 2h + 2\lambda^{-1}U(u)$ is a 2-sphere (the intersection of $Y(\lambda)$ with the 5-sphere $\{|y|^2 = 2h + 2\lambda^{-1}U(u)\}$). These 2-spheres collapse to a point as λ [redacted] $(x|x|^{-1}, (y - y_u)|y - y_u|^{-1})$, where $y_u = \omega^{-1}U(u)\, A_3(x)$. β is the desired diffeomorphism. The point y_u is the "center" of the 3-sphere of y's such that $\pi(x, y) = u$, for some x, $\beta^{-1}(u, v) = (ru, sv + y_u)$, where r and s are scalars determined by the equations

$$A(ru)(sv + y_u) = \Omega \qquad \text{and} \qquad |sv + y_u|^2 = 2h + r^{-1}U(u).$$

This completes the proof.

It remains to characterize $M^6(h, \omega)$ in the cases where $M^3(h, \omega)$ is diffeomorphic to $S^3 - 3S^1 - 2T$ or to $3(T - S^1)$. These cases occur only when $h < 0$. The following lemma is needed in the first case:

Lemma 3.8. *If* $-2h\omega^2$ *satisfies inequality B, then there exists* $\epsilon > 0$ *such that* $\pi^{-1}(X)$ *is diffeomorphic to two copies of* $S^1 \times S^1 \times D^4$, *where* $X = \{u \in \hat{S}^3 : U^2(u) + 2h\omega^2 \leqslant \epsilon\}$.

Proof. $P(X) = \{s \in \nabla : V(s) + 2h\omega^2 \leqslant \epsilon\}$. By choosing ϵ sufficiently small we can insure that $P(X)$ is an annulus in the interior of ∇ if $-2h\omega^2$ satisfies inequality B. Thus, in view of Lemma 3.3, X is diffeomorphic to two copies of $S^1 \times S^1 \times [0, 1]$. Let $Y = \{u \in M^3(h, \omega) : U^2(u) + 2h\omega^2 = 0\}$ and define $\varphi : X \to Y$ by projecting X along orbits of grad U. If $u \in X - Y$, then $\pi^{-1}(u)$ is diffeomorphic to S^3 as in the previous proposition. As u approaches Y along an orbit of the flow generated by grad U these S^3's collapse to the point $\pi^{-1}(\varphi(u))$. Hence $\varphi^{-1}(y)$ is diffeomorphic to D^4 for each

$y \in Y$. Since Y is diffeomorphic to two copies of $S^1 \times S^1$ it easily is verified that $\pi^{-1}(X)$ is diffeomorphic to two copies of $S^1 \times S^1 \times D^4$.

[redacted] *surgery consists of removing two copies of $T \times S^3$ standardly embedded in $(S^3 - 3S^1) \times S^3$ and replacing them by two copies of $\partial T \times D^4$.*

Proof. Choose $\epsilon > 0$ such that

$$\pi^{-1}(X) = \pi^{-1}(\{u \in M^3(h, \omega) : U^2(u) + 2h\omega^2 \leqslant \epsilon\})$$

is diffeomorphic to two copies of $\partial T \times D^4$. As in proposition 3.7, one shows that $\pi^{-1}(\overline{M^3(h, \omega) - X})$ is diffeomorphic to $(S^3 - 3S^1 - 2(T - \partial T)) \times S^3$, and hence, $M^6(h, \omega)$ is diffeomorphic to the space obtained from $(S^3 - 3S^1 - 2(T - \partial T)) \times S^3 \cup 2(\partial T \times D^4)$ by identifying the boundaries of these two manifolds with each other. (The boundaries in this case are equal to $2(\partial T \times S^3)$.) Thus, $M^6(h, \omega)$ is obtained from $(S^3 - 3S^1) \times S^3$ by the surgery described in the proposition.

PROPOSITION 3.10. *If $-2h\omega^2$ satisfies inequality C, then $M^6(h, \omega)$ is diffeomorphic to three copies of $\partial T \times R^4$.*

The proof of this proposition is similar to the proof of Proposition 3.9 and is omitted. It is interesting to observe that in this case $M^3(h, \omega)$ has three components each diffeomorphic to $(T - S^1)$. $M^2(h, \omega)$ also has three components which lie in disjoint neighborhoods of the corners of ∇. Hence, the shape of the triangle formed by the three bodies can not vary greatly and the same two bodies must remain close to each other relative to the third body. The motion can be approximated by considering two 2-body problems.

REFERENCES

[redacted]
1951.