J. Fluid Mech. (2021), *vol.* 928, A21, doi:10.1017/jfm.2021.803

Dynamic soliton–mean flow interaction with non-convex flux

1. Introduction

The interaction of dispersive waves with slowly varying mean flows is a fundamental and canonical problem of fluid mechanics with important applications in geophysical fluid dynamics (see, e.g. Pedlosky (2003), Mei, Stiassnie & Yue (2005), Bühler (2009) and references therein). This multiscale problem is relevant for linear or weakly nonlinear wavepackets and large amplitude solitons – in this work, we do not distinguish between solitary waves and solitons. Traditionally, the mean flow involved in the interaction is either prescribed externally, e.g. an external current, or is induced by amplitude modulations of a nonlinear wave. A different class of wave–mean flow interactions has recently been identified in Maiden *et al.* (2018), where both the dynamic mean flow and the propagating localised soliton are described by the same dispersive hydrodynamic equation, a canonical example being the Korteweg–de Vries (KdV) equation. However, the evolution of the field $u(x, t)$ occurs on two well-separated spatio-temporal scales, allowing for the distinct identification of waves and mean flows. A prototypical configuration of this (figure 1) is the propagation of a soliton through a dynamically evolving macroscopic flow, characterised by different asymptotic states $u \to u_\pm$ as $x \to \pm\infty$. We refer to such nonlinear wave interactions as soliton–mean flow interactions. The simplest mean flows are initiated by a monotone transition or step between u_- and u_+, which asymptotically develops into either a rarefaction wave (RW) or a highly oscillatory dispersive shock wave (DSW) (Gurevich & Pitaevskii 1974; El & Hoefer 2016). While the former is slowly varying, the use of the expression 'mean flow' for the latter implies some averaging over rapid oscillations. We shall refer to the step problem for dispersive hydrodynamics as a dispersive Riemann problem. Solitons, RWs and DSWs (also known as undular bores) are ubiquitous and fundamental nonlinear wave structures occurring in a variety of geophysical fluid contexts including internal waves in lakes or oceans (Boegman, Ivey & Imberger 2005; Helfrich & Melville 2006; Madsen, Fuhrman & Schäffer 2008; Jamshidi & Johnson 2020) and surface water waves (Chanson 2010; Chassagne *et al.* 2019) as well as magma and glacier flows (Scott & Stevenson 1984; Stubblefield, Spiegelman & Creyts 2020), so the problem of their interaction is of considerable interest for fluid dynamics applications. Depending upon its initial position and amplitude, the soliton may transmit or 'tunnel' through the large scale, expanding mean flow; otherwise, it remains trapped within the mean flow. Recent work has investigated the interaction between solitons and mean flows resulting from the evolution of an initial step. Both fluid conduit experiments and the theory for a rather general, single dispersive hydrodynamic conservation law were described in Maiden *et al.* (2018). A generalisation of soliton–mean flow interaction to the bidirectional case for a pair of conservation laws described by the defocusing nonlinear Schrödinger equation (NLS) equation was explored in Sprenger, Hoefer & El (2018). Soliton–mean flow interaction in the focusing NLS equation was investigated in Biondini & Lottes (2019). A similar problem involving the interaction of linear wavepackets with shallow-water wave mean flows modelled by the KdV equation was studied using an analogous modulation theory framework in Congy, El & Hoefer (2019). Aside from the focusing NLS case, for which mean flow evolution is described by an elliptic system of equations, and the present work, the models previously investigated in the context of soliton–mean flow interaction were limited to dispersive conservation laws with hyperbolic, convex flux.

The focus of this work is the study of soliton–mean flow interaction when the governing dispersive hydrodynamics exhibits a non-convex hydrodynamic flux. As we show, the presence of non-convex flux, e.g. the cubic flux in the modified KdV (mKdV) equation

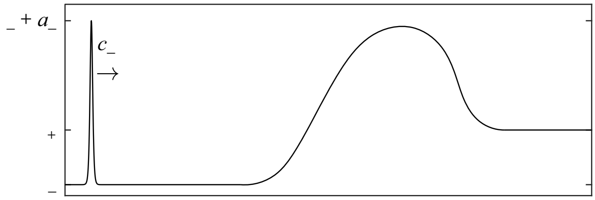


Figure 1. Representative initial configuration for soliton–mean flow interaction. The narrow soliton with amplitude a_- on the uniform mean flow $\overline{u}_-$ transmits through the broad hydrodynamic flow if it reaches the uniform mean flow $\overline{u}_+$, surpassing all initial mean-generated oscillations. Otherwise, it experiences trapping inside the mean flow. The mean flow generally exhibits expansion and compression waves.

trapping scenarios realised in the KdV case. First of all, due to the non-convex flux, the mKdV equation supports a much broader family of solitons and mean flow solutions than the KdV equation, including localised solutions in the form of exponentially decaying solitons of both polarities and, depending on the dispersion sign, kinks and algebraic solitons. The mKdV non-convex mean flow features include undercompressive DSWs (an alternative interpretation of kinks), contact DSWs (CDSWs) and compound two-wave structures (Kamchatnov *et al.* 2012, 2013; El, Hoefer & Shearer 2017). Here, we investigate how the solution features that arise due to non-convex flux affect soliton–mean flow interactions. In particular, we show that soliton transmission for the defocusing mKdV equation can be accompanied by a soliton polarity change. In the focusing case, there is a soliton–mean flow interaction in which an exponential soliton is asymptotically transformed into a trapped algebraic soliton. These are just two examples of the rich catalogue of soliton–mean flow interactions we describe in this paper.

Key to the study of soliton–mean flow interaction is scale separation, whereby the characteristic length and time scales of the propagating soliton are much shorter than those of the mean flow. The rapidly oscillating structure of dispersive hydrodynamic flows motivates the use of multiscale asymptotic methods. Here, we will make extensive use of one such method known as Whitham modulation theory (Whitham 1974), which is based on a projection of the scalar dispersive hydrodynamics onto a three-parameter family of slowly varying periodic travelling wave solutions to the governing equation. The projection is achieved, equivalently, by averaging conservation laws, an averaged variational principle, or multiple scale perturbation methods. The dispersive hydrodynamics is then approximately described by a system of three first-order quasilinear partial differential equations (PDEs) – the Whitham modulation equations – for the periodic travelling wave's parameters such as the wave amplitude, the wavenumber and the period mean. Within the framework of Whitham modulation theory, the original dispersive Riemann problem is posed as a special Riemann problem, sometimes called the Gurevich–Pitaevskii (GP) problem (Gurevich & Pitaevskii 1974), for the modulation equations subject to piecewise constant initial data with a single discontinuity at the origin. Continuous, self-similar solutions of the GP problem describe RW and DSW mean flow modulations.

Classical DSW modulation theory has been developed for the KdV equation (Gurevich & Pitaevskii 1974) and other 'KdV-like' equations, both integrable and non-integrable (El 2005; El & Hoefer 2016). It is useful to identify this class of KdV-type equations, or classical, convex dispersive hydrodynamic equations, as those equations for which the associated Whitham modulation equations are strictly hyperbolic and genuinely nonlinear. In this case, the generic solution of the GP problem is either a DSW or a RW. More broadly, even non equations remain strictly hyperbolic and genuinely nonlinear. Therefore, we shall call the DSWs generated within the framework of convex dispersive hydrodynamics convex DSWs.

It was shown in Maiden *et al.* (2018) that the interaction of a soliton with a RW is described by an exact, soliton limit reduction of the Whitham modulation system, which we call the solitonic modulation system. Two integrals or adiabatic invariants of the solitonic modulation system were identified that determine the amplitude and phase shift of the soliton when transmitted through the variable mean flow. The non-existence of a transmitted soliton (zero or negative transmitted amplitude) signifies soliton trapping within the mean flow. The soliton–DSW transmission/trapping conditions were shown to be equivalent to those for the soliton–RW interaction by the fundamental property of hydrodynamic reciprocity of the modulation solution, which is related to time reversibility of the original dispersive hydrodynamics.

In this paper, we investigate the effects of a flux's non-convex with

$\Omega(k,\bar{u}) = o(k)$ as $k \to 0$ and $\Omega_{kk}(k,\bar{u})$ is not identically zero in order to separate the long-wave hydrodynamic flux from short-wave dispersive effects. The field of dispersive hydrodynamics encompasses multiscale nonlinear wave solutions of initial and boundary value problems for (2.1) (possibly with perturbations) in which at least two length and time scales play a prominent role: the oscillatory scale (e.g. the width of a soliton or the wavelength/period of a periodic travelling wave) and a longer, hydrodynamic scale (e.g. the slowly varying oscillatory amplitude of a wavepacket or DSW). One canonical dispersive hydrodynamic problem for (2.1) is the so-called GP problem (Gurevich & Pitaevskii 1974) in which $u(x,0)$ for $x \in \mathbb{R}$ exhibits a sharp, monotone transition between two distinct far-field boundary conditions. The solution of the GP problem then describes the long-time asymptotic behaviour for more general initial data with distinct far-field equilibrium states.

When f

system is non-strictly hyperbolic (the eigenvectors $\boldsymbol{r}_i$ span $\mathbb{R}^3$ but multiple eigenvalues are admissible), then it is not genuinely nonlinear either (Dafermos 2016). The converse is generally not true. Nevertheless, a non-convex system can exhibit convex properties in a restricted domain $\mathcal{D} \subset \mathcal{A}$.

The KdV–Whitham modulation system (3.1) is strictly hyperbolic and genuinely nonlinear for all admissible $\boldsymbol{u} \in \mathcal{A}$ (Levermore 1988), while for the mKdV equation, the properties of strict hyperbolicity and genuine nonlinearity depend on the sign of μ and on $\boldsymbol{u}$ (El *et al.* 2017).

An important ingredient for modulation theory is the equation for k in (3.1)

$$k_t + [\omega(\bar{u}, k, a)]_x = 0, \tag{3.3}$$

known as the conservation of waves, where $\omega(\bar{u}, k, a)$ is the travelling wave frequency.

Soliton–mean interaction theory is based on the fundamental property of Whitham modulation systems that we postulate here in a general form and later explicitly justify for mKdV: in the $k \to 0$ soliton limit, the modulation system (3.1) admits the following exact reduction (Gurevich, Krylov & El 1990):

$$\begin{bmatrix} \bar{u} \\ a \end{bmatrix}_t + \begin{bmatrix} f'(\bar{u}) & 0 \\ g(a, \bar{u}) & c(a, \bar{u}) \end{bmatrix} \begin{bmatrix} \bar{u} \\ a \end{bmatrix}_x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{3.4}$$

where $c(a, \bar{u}) = \lim_{k\to 0}(\omega/k)$ is the soliton amplitude–speed relation for propagation on the background $\bar{u}$ and $g(a, \bar{u})$ is a coupling function that is system dependent. Equation (3.4) is called the solitonic modulation system.

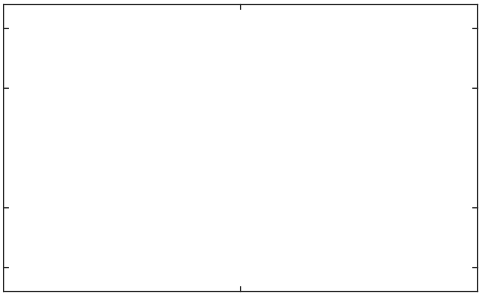
The third modulation equation (3.3) is identically satisfied for $k = 0$ while for $0 < k \ll 1$, it assumes at leading order the form

$$k_t + [c(a, \bar{u})k]_x = 0. \tag{3.5}$$

Equation (3.5) can be added to the solitonic modulation system (3.4) to give an approximate modulation system for a train of non-interacting solitons propagating on a variable mean flow. Equation (3.5) then signifies the conservation of the number of solitons in the train. We shall refer to the combined system (3.4) and (3.5) as the augmented solitonic modulation system. Note that a particular case of this system was derived in Grimshaw (1979) for slowly varying soliton solutions of the variable coefficient KdV equation.

The soliton train interpretation of the modulation system (3.4) is instrumental for a solitonic dispersive hydrodynamics as it enables the description of a single modulated soliton by treating the soliton amplitude $a(x, t)$ as a spatio-temporal field, in contrast to standard soliton perturbation theory where the soliton's parameters evolve temporally along its trajectory in the x, t-plane; see, e.g. Kivshar & Malomed (1989). Additionally, as we will show, the introduction of the fictitious wavenumber field $k(x, t)$

a convenient normalisation is suggested by the requirement to maintain strict hyperbolicity of the solitonic system in the limit of vanishing amplitude where the long-wave speed $f'(\bar{u})$ and soliton speed $c(a, \bar{u})$ must coincide. The variable $\tilde{k}$ can be identified as an amplitude-type variable (El 2005), so that $\tilde{k} = 0 \iff a = 0$, and requires that the hydrodynamic and solitonic Riemann invariants coincide when $\tilde{k} \to 0$, i.e. $Q(0, \bar{u}) = \bar{u}$. As a result, the system (3.11) reduces to a single hyperbolic equation $\bar{u}_t + f'(\bar{u})\bar{u}_x = 0$. The situation is different for non-convex systems, where two or more distinct Riemann invariants associated with the same characteristic speed may exist. For example, for cubic flux $f(\bar{u}) = \bar{u}^3$, the mean flow equation $\bar{u}_t + 3\bar{u}^2\bar{u}_x = 0$ is invariant with respect to the transformation $\bar{u} \to -\bar{u}$ so another possible normalisation is $Q(0, \bar{u}) = -\bar{u}$. To avoid ambiguity, we will be using the



(3.21*a*,*b*), yielding the relations between admissible values of $a_\pm$ and $k_\pm$ in (3.23*a*,*b*).

conditions (3.24), (3.25). Otherwise, if the transmitted amplitude $a_- \leq 0$, the soliton remains trapped within the mean flow.

The generalisation to negative (dark) soliton interaction with mean flow is straightforward. For this, it is convenient to introduce a signed amplitude a, which enables the representation of both bright $a > 0$ and dark $a < 0$ solitons. Assuming negative initial amplitude $a_\pm < 0$, forward/backward transmission requires that the transmitted amplitude $a_\mp$ maintains the same, negative, sign. Generally, the condition $a_+ a_- > 0$ is the sufficient condition for transmission in both bright and dark soliton cases. Its negation implies trapping.

In all cases of forward/backward transmission/trapping, the soliton trajectory for $t > 0$ is given by the characteristic,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = c(a(x,t), \bar{u}(x,t)), \quad x(0) = x_0, \tag{3.28a,b}$$

where $|x_0| \gg 1$ so that the soliton is initially well separated from the initial step in the mean flow at $x = 0$.

In the present work, we consider the implications of a non-convex solitonic modulation system (3.4) on the above soliton transmission and trapping scenarios. As described in § 3.1, non-convexity enters when strict hyperbolicity and/or genuine nonlinearity is lost via one of the three conditions: $f''(\bar{u}) = 0$, $f'(\bar{u}) = c(a, \bar{u})$ or $c_a(a, \bar{u}) = 0$ for any $(\bar{u}, a) \in \mathcal{A}_0$.

In Maiden *et al.* (2018), positivity of the transmitted amplitude (one of $a_\pm$) was proposed

3.3. *Hydrodynamic reciprocity*

So far, we have assumed that the mean flow satisfies the simple wave equation $\bar{u}_t + f'(\bar{u})\bar{u}_x = 0$. For step initial data (3.22), the only candidate continuous solution is a RW

$$\bar{u}(x,t) = \begin{cases} u_- & x < f'(u_-)t, \\ (f')^{-1}(x/t) & f'(u_-)t < x < f'(u_+)t, \\ u_+ & f'(u_+)t < x, \end{cases} \tag{3.30}$$

so long as the admissibility criterion $f'(u_-) < f'(u_+)$ holds, corresponding to expansive initial data. As will be shown in the next section, there is a much richer variety of dispersive mean flows generated by the mKdV GP problem when the initial data are compressive. Thus, we need soliton–mean flow modulation theory to be flexible enough to accommodate a wide class of mean flows.

The solitonic modulation equations (3.4), (3.5) directly apply for expansive mean flow initial data, yielding a description of soliton–RW interaction. For compressive initial data (3.22), rather than form a discontinuous shock solution, a DSW is formed that occupies the space–time region $A \subset \mathbb{R} \times (0, \infty)$ where the solution is described by the full system of Whitham modulation equations for a slowly varying nonlinear periodic wave. As a result, the Riemann invariant q and secondary invariant kp of the augmented solitonic system (3.4), (3.5) are not conserved in A, and our arguments leading to the transmission and phase conditions (3.24), (3.25) do not apply to the soliton interaction with the DSW mean flow.

To address this, we invoke an important property of the dispersive conservation law (2.1): time reversibility. A consequence of time reversibility is the continuity of the modulation solution for all $(x, t) \in \mathbb{R}^2$. For compressive data, we consider the solution for $t < 0$ that consists of a simple wave described by (3.30), i.e. the expansive mean flow case. Then, since q and kp are constant for all $x \in \mathbb{R}$ and $t < 0$, they remain constant by continuity for (x, t) in the complement of A, outside of the oscillatory region, where the augmented solitonic system (3.4), (3.5) remains valid. Note that for the Riemann data (3.22), (3.23*a*,*b*), the solution remains continuous outside $\mathbb{R}^2 \setminus \{(0, 0)\}$, which is justified by taking the limit of smooth solutions. This property was called hydrodynamic reciprocity in Maiden *et al.* (2018) and has been used previously in the characterisation of DSWs for a single or pair of dispersive hydrodynamic conservation laws (El 2005). Since the transmission and phase conditions (3.24), (3.25) hold outside the oscillatory region, hydrodynamic reciprocity allows us to predict the transmitted amplitude and phase shift Δ of a soliton interacting with DSW mean flows entirely within the framework of the augmented solitonic modulation system (3.4), (3.5).

The details of the modulation dynamics for the soliton within the interior of the oscillatory region A can, in principle, be described by a degenerate two-phase solution (see Flaschka, Forest & McLaughlin (1980) for multiphase modulation theory of the KdV equation). However, as we will show, this rather technical approach can be partially, approximately circumvented by replacing $f(\bar{u})$ in the characteristic equation (3.13) by an appropriate choice of the mean flow variation and effectively defining a new adiabatic invariant q holding within A.

4. Modulation theory for the mKdV equation

As the simplest example of dispersive hydrodynamics with non-convex flux, we study the mKdV equation (2.3). The mean flow behaviours that arise when solving (2.3) subject to (3.22) depend on the sign of the dispersive term $\mathrm{sgn}(\mu)$. The mKdV hyperbolic flux

$f(u) = u^3$ exhibits the inflection point $f''(0) = 0$ so that non-convexity affects the solutions whenever the initial data contain an open interval including the point $u = 0$. For either sign of μ, the mKdV equation allows for solitons of both polarities by the symmetry $u \to -u$. The linear dispersion relation is

$$\omega_0 = 3\bar{u}^2 k + \mu k^3. \tag{4.1}$$

The purpose of this section is twofold: (i) to obtain the augmented solitonic modulation system (3.4), (3.5) by direct computation for the mKdV–Whitham system and (ii) to explore the implications of the mKdV's non-convex flux for the structure of the augmented solitonic modulation system. But first, we need to understand mKdV's travelling wave solutions. In order to be self-contained, Appendix A presents a compendium of the results on mKdV travelling wave solutions from El *et al.* (2017) necessary for the development in this paper, which we briefly summarise. The mKdV equation differs from the KdV equation in that it supports solitons of both polarities for either sign of the dispersion μ. For $\mu > 0$, bright soliton solutions occur when $u_1 \to u_2$ and dark soliton solutions occur when $u_3 \to u_4$. For $\mu < 0$, solitons arise when $u_2 \to u_3$ with bright solitons as solutions between u_3 and u_4 while dark solitons occur between u_1 and u_2. The amplitude–speed relations (A6) and (A8) for bright and dark exponential solitons, respectively, can be combined into a single relation by introducing the convention that $a > 0$ for bright solitons and $a < 0$ for dark solitons. Then, the general formula

$$c(a, \bar{u}) = \tfrac{1}{2}a^2 + 2a\bar{u} + 3\bar{u}^2, \quad a \in \mathbb{R} \tag{4.2}$$

holds, covering all cases: $\mu \lessgtr 0$, dark and bright exponential solitons. Note that this formula also includes kinks ($a = -2\bar{u}$, $c = \bar{u}^2$, $\mu > 0$) and algebraic solitons ($a = -4\bar{u}$, $c = 3\bar{u}^2$, $\mu < 0$). From now on, we will be assuming the generalised amplitude $a \in \mathbb{R}$.

The system of modulation equations for the mKdV equation (2.3) was first derived in Driscoll & O'Neil (1975) following Whitham's original averaging procedure (Whitham 1965), and reduced to diagonal form.

A derivation of the travelling wave solutions and the respective modulation equations for the Gardner equation (an extended version of mKdV), revealing the differences between various modulationally stable DSW structures arising in the $\mu > 0$ and $\mu < 0$ cases was performed in Kamchatnov *et al.* (2012) and then utilised in El *et al.* (2017) for the analysis of modulated mKdV solutions in the zero-viscosity limit of the mKdV–Burgers equation. Following El *et al.* (2017), the mKdV modulation system is

$$\frac{\partial \lambda_i}{\partial t} + W_i(\lambda)\frac{\partial \lambda_i}{\partial x} = 0, \quad i = 1, 2, 3, \tag{4.3}$$

where λ_i are Riemann invariants related to roots of the potential function $Q(\bar{u})$,

$$\lambda_1 = \tfrac{1}{2}(u_1 + u_2), \quad \lambda_2 = \tfrac{1}{2}(u_1 + u_3), \quad \lambda_3 = \tfrac{1}{2}(u_2 + u_3). \tag{4.4a–c}$$

The characteristic velocities W_i are given in Appendix B.

Applying the limit $\lambda_2 \to \lambda_3$ to (4.3) and using (B5), (B6) gives the reduced diagonal system

$$\left.\begin{aligned} \frac{\partial \lambda_1}{\partial t} + 3\lambda_1^2 \frac{\partial \lambda_1}{\partial x} &= 0, \\ \frac{\partial \lambda_3}{\partial t} + (\lambda_1^2 + 2\lambda_3^2)\frac{\partial \lambda_3}{\partial x} &= 0. \end{aligned}\right\} \tag{4.5}$$

Using $\bar{u} = u_1 = \lambda_1$ and $a = u_3 - u_1 = 2(\lambda_3 - \lambda_1)$ (see (A5)), we can now write (4.5) as

$$\left.\begin{aligned} \bar{u}_t + 3\bar{u}^2\bar{u}_x = 0, \\ a_t + \left(\tfrac{1}{2}a^2 + 2a\bar{u} + 3\bar{u}^2\right)a_x + (a^2 + 4a\bar{u})\bar{u}_x = 0. \end{aligned}\right\} \tag{4.6}$$

The system (4.6

result with $q^2 = \lambda_3^2$. The approximate conservation of waves equation (4.9) is subject to corrections of order $k\,\mathrm{e}^{-\alpha k}$ where $\alpha = \pi\sqrt{2(\lambda_1^2 - \lambda_3^2)} = \pi\sqrt{-a(a+4\bar{u})/2}$ as $k \to 0$.

The solitonic modulation system (4.8) loses strict hyperbolicity when $3\bar{u}^2 = \bar{u}^2 + 2q^2$ – corresponding to $f'(\bar{u}) = c(a, \bar{u})$ in the notation of (3.4) – which yields $q^2 = \bar{u}^2$, and implying via (4.7) either

$$a = 0, \quad \text{or} \quad a = -4\bar{u}. \tag{4.11}$$

As mentioned earlier, the $a = 0$ case corresponds to a reduction in order of the solitonic modulation system (4.8) to the mean flow equation $\bar{u}_t + 3\bar{u}^2\bar{u}_x = 0$. Strictly speaking, it does not correspond to the loss of strict hyperbolicity as traditionally defined for Whitham modulation systems, but it is relevant for the general tunnelling conditions (3.29*a–c*).

Genuine nonlinearity is lost when (4.11) holds or, alternatively, if $f''(\bar{u}) = 0$, or $c_a = 0$, cf (3.9), (3.10), i.e.

$$\bar{u} = 0 \quad \text{or} \quad a = -2\bar{u} \quad \Leftrightarrow \quad q = 0. \tag{4.12}$$

In all cases, the soliton speed in terms of the Riemann invariants is given by

$$C(q, \bar{u}) = \bar{u}^2 + 2q^2 > 0, \quad \text{for } a \neq 0. \tag{4.13}$$

As shown in § 3, for modulations with constant q, the wave conservation equation (4.9) is diagonalised by the variable kp, where $p(q, \bar{u})$ is given by (3.16). Using (4.13) and $f'(u) = 3u^2$ in (3.16), we determine $p(q, \bar{u})$ for mKdV solitonic modulations,

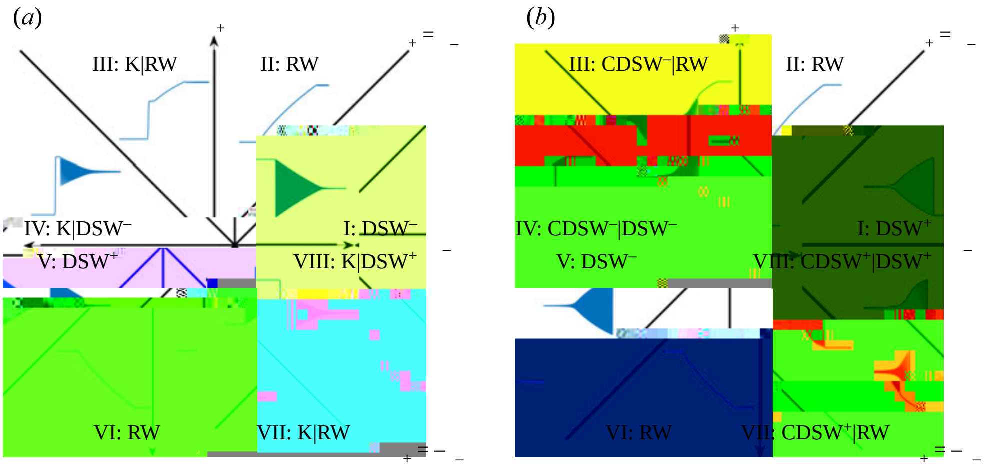
$$\begin{aligned} p(q, \bar{u}) &= \exp\left(-\int_{\bar{u}_0}^{\bar{u}} \frac{u}{u^2 - q^2}\,\mathrm{d}u\right) \\ &= |q^2 - \bar{u}^2|^{-1/2}, \quad q^2 \neq \bar{u}^2, \end{aligned} \tag{4.14}$$

where we have chosen $\bar{u}_0^2 = q^2 + 1$ for convenience.

5. Classification of mean flows in the mKdV GP problem

The solution to the GP problem for mKdV was classified in El *et al.* (2017) by combining previous work on the Riemann problem for either sign of dispersion (Chanteur & Raadu 1987; Marchant 2008) and elaborating on the GP problem classification for the Gardner equation $u_t + 6uu_x - 6\alpha u^2 u_x + u_{xxx} = 0$ (Kamchatnov *et al.* 2012). The wave behaviour that emerges from the GP problem depends on the sign of μ and relative sign and magnitude of u_- and u_+, as shown in the classification diagram of figure 3. We refer to the octants in this figure as regions I to VIII, counted in a counterclockwise fashion. Owing to its universality as a model of weakly nonlinear, long dispersive waves (El *et al.* 2017), the mKdV equation provides a fundamental description of the GP problem for other PDEs with non-convex flux.

RWs and DSWs solve the GP problem in certain convex and non-convex cases. DSWs are classified as DSW$^+$ and DSW$^-$ according to the polarity of the solitary wave generated at one of the edges – leading or trailing, depending on the DSW orientation. In the non-convex case, we see the emergence of additional wave structures. These occur when the hydrodynamic flux $f(u) = u^3$ exhibits an inflection point $u = 0$ within the range of step data (2.4) so that $u_+u_- < 0$. Particularly, when $\mu > 0$, and $u_- = -u_+$, the long-time asymptotic solution is a kink, which is an undercompressive shock in the limit $\mu \to 0^+$. When $\mu < 0$ and $u_- = -u_+$, the long-time asymptotic solution is a CDSW whose leading,


(a)
III: K|RW
II: RW
IV: K|DSW⁻
I: DSW⁻
V: DSW⁺
VIII: K|DSW⁺
VI: RW
VII: K|RW
(b)
III: CDSW⁻|RW
II: RW
IV: CDSW⁻|DSW⁻
I: DSW⁺
V: DSW⁻
VIII: CDSW⁺|DSW⁺
VI: RW
VII: CDSW⁺|RW

For both signs of μ, the transmission and phase conditions can be determined from (3.24), (3.25), (4.7), (4.14) as

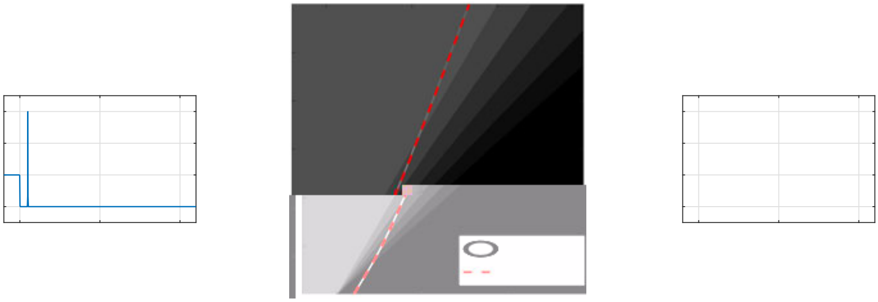
$$\frac{a_+}{2} + u_+ = \frac{a_-}{2} + u_-, \quad \frac{k_-}{k_+} = \sqrt{\frac{q_-^2 - u_-^2}{q_+^2 - u_+^2}} = \sqrt{\frac{\frac{1}{4}a_-^2 + a_- u_-}{\frac{1}{4}a_+^2 + a_+ u_+}}. \qquad (6.1a,b)$$

Notably, these transmission and phase conditions are exactly the same as those for the KdV equation $u_t + (u^2)_x = \mu u_{xxx}$ with convex flux (Maiden *et al.* 2018). Although, for mKdV, the conditions apply for both positive and negative soliton amplitudes.

The tunnelling condition (3.29*a–c*) fails when the characteristic speeds $f'(\bar{u})$ and $C(q, \bar{u})$ cross, which occurs when (see (4.11))

$$q^2 = \bar{u}^2 \implies a \in \{0, -4\bar{u}\}. \qquad (6.2)$$

Crossing through $a = 0$ gives the same condition as in the convex case, where for bright solitons, $a > 0$ on the transmitted side implies tunnelling, and $a \leq 0$ means the soliton is trapped. For dark solitons, the inequalities must be reversed.



Dispersion	Direction	Region II - RW ($u_+ > u_- > 0$)	Region VI - RW ($u_+ < u_- < 0$)
$\mu > 0$	$R \to L$	No bright soliton solutions	Tunnelling if $a_+ > a_{cr} = 2(u_- - u_+)$
$\mu < 0$	$L \to R$	Tunnelling if $a_- > a_{cr} = 2(u_+ - u_-)$	Tunnelling if $a_- > a_{cr} = -2(u_+ + u_-)$

Table 1. Results for the bright soliton tunnelling problem through RWs; $R \to L$ means that $x_0 = x_+$ and the soliton propagates from right (R) to left (L), otherwise $x_0 = x_-$ ($L \to R$).

where D and E are obtained by continuity of $x(t)$

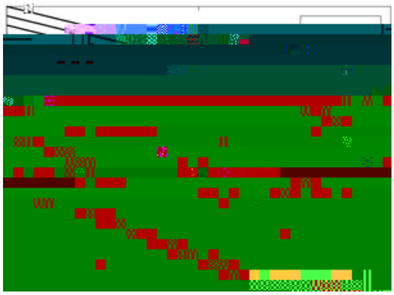
$$D = \tfrac{3}{2}x_+^{2/3}(2u_+^2 - 2q^2)^{1/3} \tag{7.4}$$

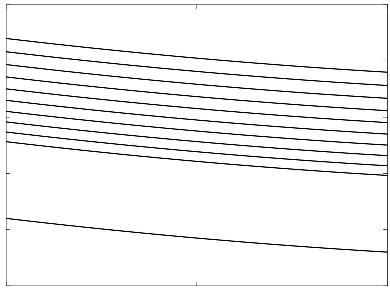
$$E = x_+\sqrt{\frac{u_+^2 - q^2}{u_-^2 - q^2}}. \tag{7.5}$$

The phase shift is $\Delta = E - x_+$, which matches the condition given by (6.1*a*,*b*).

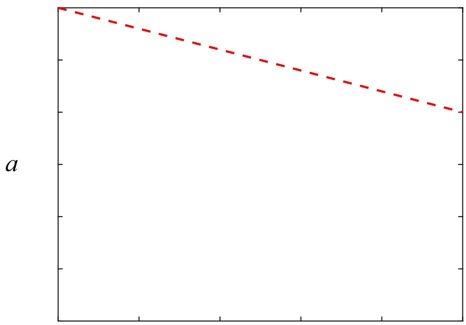
A similar analysis can be carried out for each region in figure 3 to determine the tunnelling criterion. We summarise the remaining results without detailing the analysis for each case in table 1 for either sign of μ in regions II and VI. Note that for $\mu < 0$ in region VI, the tunnelling criterion is different than the condition that $a_+ > 0$. This is because there are cases for valid initial soliton amplitudes a_- where the amplitude crosses $-4\bar{u}$ during interaction with the RW, causing the soliton to become trapped. In the limit $t \to \infty$, the trapped soliton limits to an algebraic soliton moving with the mean flow velocity $3\bar{u}^2$.

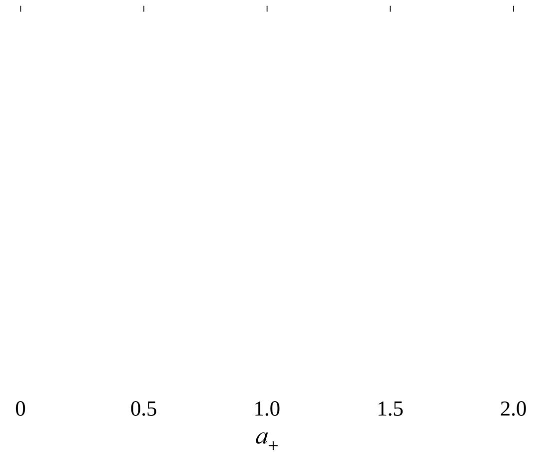
Figure 5 illustrates the loss of strict hyperbolicity when $\mu < 0$ for non-zero amplitudes by depicting the wave curves $a(\bar{u})$ correspondi24.1(alid)-34TJ(93Tf.7795[(zer)8.1(o)-268.215-1.1178TD

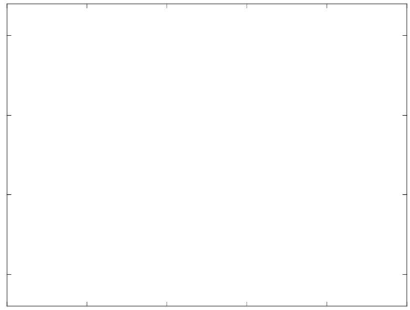




u_+	u_-	a_+	T_{final}	a_- (pred)	a_- (num)	$\Delta x/x_-$ (pred)	$\Delta x/x_-$ (num)


a


0
0.5
1.0
1.5
2.0
a_+



8.1. *Soliton–kink interaction*

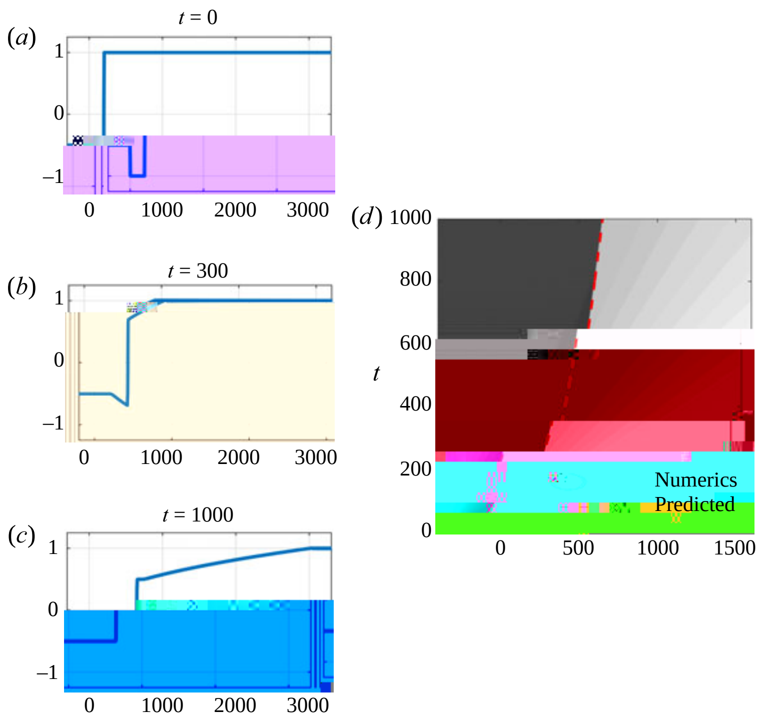
A kink solution to the GP problem when $\mu > 0$ is realised when $u_+ = -u_-$. To be definite, we assume that $u_- < 0$. The kink velocity $u_-^2 = u_+^2$ is slower than the soliton velocity for any amplitude so interaction happens from left to right with $x_0 = x_- < 0$. By the soliton existence conditions (A7), when $u_- < 0$, we must initialise with a bright soliton $(a_- > 0)$ on the left side. Since bright solitons cannot exist on the right side of the kink where $u_+ > 0$, we expect that the soliton polarity undergoes a switch as a result of kink interaction in order for the soliton to be a valid solution. To determine the transmitted soliton amplitude, we observe that, under the quadratic transformation (B3), the mKdV

u_-	u_+	a_-	t_{final}	a_+	a_+ (num)	Δx	Δx (num)	$\Delta x/\lvert x_-\rvert$ (num)	Δx_{kink}
−1	1	0.5	200	−0.5	−0.4991	0	1.6530	0.0331	−2.3438
−1	1	1.0	200	−1.0	−1.0000	0	2.1484	0.0430	−3.8086
−1	1	1.5	500	−1.5	−1.4999	0	3.0273	0.0605	−5.8594
−2	2	1.5	50	−1.5	−1.4988	0	0.9766	0.0195	−1.6113

Direction	Region III - kink|RW ($u_+ > -u_- > 0$)	Region IV - kink|DSW ($u_- < -u_+ < 0$)	Region VII - kink|RW ($u_+ < -u_- < 0$)	Region VIII - kink|DSW ($-u_- < u_+ < 0$)
$R \to L$	No soliton solutions	No soliton solutions	Tunnelling through RW if $a_+ > -2u_- - 2u_+$, trapped to the right of the kink	Tunnelling through DSW always, trapped to the right of the kink
$L \to R$	Tunnelling through kink, polarity flips, trapped to the left of the RW	Tunnelling through kink, polarity flips, trapping in DSW if $a_- < 2u_+ - 2u_-$	No soliton solutions	No soliton solutions

Table 7. Results for $\mu > 0$ with bright solitons interacting with hybrid mean flows.

<table>
<tr><th>Direction</th><th>Region III - CDSW|RW
($u_+ > -u_- > 0$)</th><th>Region IV - CDSW|DSW
($u_- < -u_+ < 0$)</th><th>Region VII - CDSW|RW
(u_+ ĕ</th></tr>
</table>


(a)
t = 0
1
0
−1
0
1000
2000
3000
(b)
t = 300
1
0
−1
0
1000
2000
3000
(c)
t = 1000
1
0
−1
0
1000
2000
3000
(d)
1000
800
600
400
200
0
t
0
500
1000
1500
Numerics
Predicted

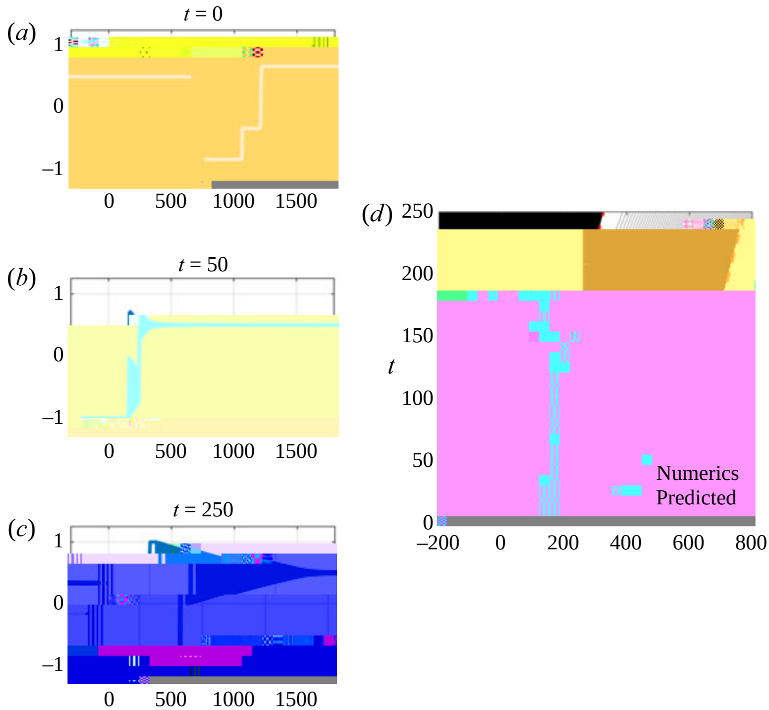


Figure 14. Kink–DSW interaction with $\mu = 1$, $a_+ = 1$; $x_+ = 150$, $u_- = -1$ and $u_+ = -0.5$. (*a*) Initial condition. (*b*) Configuration at intermediate time $t = 50$. (*c*) Configuration at $t = 250$. (*d*) Space–time contour plot of solution with kink characteristic (dashed). The predicted a_- is 2 and the predicted Δ is -75. The numerical solution gives $a_- = 2.0022$ and $\Delta = -74.56$ at $t = 250$.

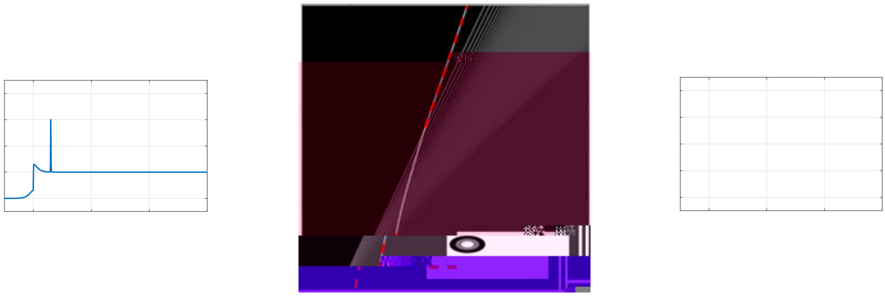
Again for DSWs, interaction with the kink causes the DSW to switch polarity as seen in the numerical experiment of figure 14. These polarity switches are only possible due to non-convexity. The kink–DSW trajectory is given by (9.1*a*,*b*), where the DSW mean flow $\bar{u} = \bar{u}(x/t)$ is determined by (5.4), (5.2*a*–*c*) and (5.3*a*,*b*).

Note that kink–kink interaction is not possible as multiple kinks will co-propagate.

10. Generalisation to arbitrary soliton–convex mean flows

We have described soliton tunnelling interactions specifically with mean flows that emerge from a Riemann step-type initial condition. However, the tunnelling problem can be generalised to determine the phase shift and amplitude of a soliton that tunnels through an arbitrary mean hydrodynamic flow. If tunnelling occurs, only the far-field mean flow conditions u_- and u_+ are needed to predict the transmitted soliton amplitude. The phase shift can be calculated by approximating the initial mean flow $\bar{u}(x, 0)$ with a series of step functions and taking a limit that results in the Riemann integral

$$\Delta = \int$$



(la Forgia *et al.* 2020). When reflections occur due to additional topographic features, such large-scale mean flows may encounter solitons of different speeds (la Forgia *et al.* 2020), leading to the type of soliton–mean interaction described here. Along with the mKdV equation, internal waves can be modelled by the Gardner equation that combines KdV and mKdV hydrodynamic fluxes (Grimshaw 2002; Helfrich & Melville 2006). The generalisation of our results to the Gardner equation is straightforward. We stress that our approach does not make use of the integrability of the mKdV equation, so can be applied to a non-integrable, non-convex dispersive hydrodynamics. In particular, a new, intriguing non-convex scalar model has recently been derived for the contour dynamics [illegible]

Appendix A. The mKdV travelling wave solutions

The mKdV travelling wave solutions $u = u(\eta), \eta =$

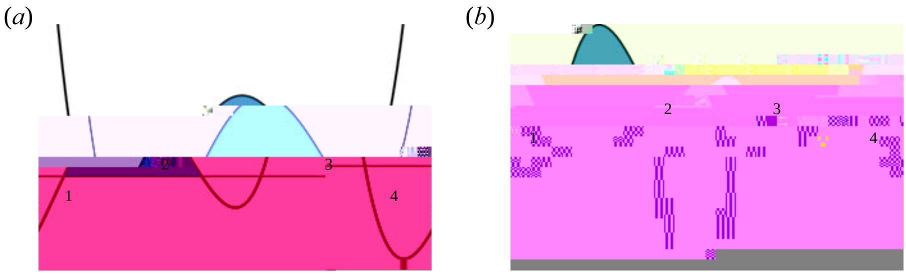


Figure 16. Potential curve $Q(u)$ of the nonlinear oscillator equation (A1). Travelling wave solutions exist in the shaded regions. (a) $\mu > 0$ and (b) $\mu < 0$.

with speed $U = \bar{u}^2$, which matches the classical shock speed determined by the Rankine–Hugoniot condition.

(ii) For $\mu < 0$, travelling wave solutions can occur between u_1 and u_2 or between u_3 and u_4. Between u_3 and u_4, the travelling wave solution is

$$u = u_3 + \frac{(u_4 - u_3)\mathrm{cn}^2(\theta, m)}{1 + \dfrac{u_4 - u_3}{u_3 - u_1}\mathrm{sn}^2(\theta, m)}, \tag{A11}$$

with $m = m_- = (u_4 - u_3)(u_2 - u_1)/(u_4 - u_2)(u_3 - u_1)$. The wavenumber is given by the same formula (A4). When $u_3 \to u_2$ ($m_- \to 1$) the solution becomes a bright exponential soliton with amplitude $a = u_4 - u_2$ and background $\bar{u} = u_2$

$$u = u_2 + \frac{u_4 - u_2}{\cosh^2\theta + \dfrac{u_4 - u_2}{u_2 - u_1}\sinh^2\theta}. \tag{A12}$$

This soliton solution travels according to the same soliton amplitude–speed relation (A6) as in the case $\mu > 0$. Due to the root ordering, valid bright soliton amplitudes for the solution to exist are given by

$$a > max(0, -4\bar{u}), \tag{A13}$$

with no constraint on the background $\bar{u}$.

For $\mu < 0$, there is a special type of travelling wave solution expressible in terms of trigonometric functions. Again, these solutions occur either between u_1 and u_2 or between u_3 and u_4 but under the additional constraint that $u_3 = u_4$ in the first case and $u_1 = u_2$ in the second case. For $u_3 \le u \le u_4$, $u_1 = u_2$ the solution is given by

$$u = u_3 + \frac{u_4 - u_3}{1 + \dfrac{u_4 - u_1}{u_3 - u_1}\tan^2\theta}. \tag{A14}$$

The nonlinear trigonometric solution (A14) has no analogue in KdV theory. When $u_3 \to u_2 = u_1 \equiv \bar{u}$, the solution (A14) becomes an algebraic bright soliton described by

$$u = u_1 + \frac{u_4 - u_1}{1 + (u_4 - u_1)^2\eta^2/4}, \tag{A15}$$

with amplitude $a = u_4 - u_1 = -4\bar{u}$ and travelling at speed $U = 3u_1^2 = 3\bar{u}^2$, which is the

The solution oscillating between u_1 and u_2

EL, G.A. & HOEFER, M.A. 2016 Dispersive shock waves and modulation theory. *Physica* D 333, 11–65.

EL, G.A. & SMYTH, N.F. 2016 Radiating dispersive shock waves in non-local optical media. *Proc. R. Soc.* A 472 (2187), 20150633.

FLASCHKA, H., FOREST, M.G. & MCLAUGHLIN, D.W. 1980 Multiphase averaging and the inverse spectral solution of the Korteweg-de Vries equation. *Commun. Pure Appl. Maths* 33, 739–784.

LA FORGIA, G., OTTOLENGHI, L., ADDUCE, C. & FALCINI, F. 2020 Intrusions and solitons: propagation and collision dynamics. *Phys. Fluids* 32 (7), 076605.

GRIMSHAW, R. 1979 Slowly varying solitary waves. I. Korteweg-de Vries equation. *Proc. R. Soc. Lond.* A 368, 359–375.

GRIMSHAW, R.H.J. 2002 Internal solitary waves. In *Environmental Stratified Flows* (ed. R.H.J. Grimshaw), pp. 1–27. Kluwer.

GRUZINOV, A.V. 1992 Contour dynamics of the Hasagawa-Mima equation. *JETP Lett.* 55 (1), 76–80.

GUREVICH, A.V., KRYLOV, A.L. & EL, G.A. 1990 Nonlinear modulated waves in dispersive hydrodynamics. *Sov. Phys. JETP* 71 (5), 899–910.

GUREVICH, A.V. & PITAEVSKII, L.P. 1974 Nonstationary structure of a collisionless shock wave. *Sov. Phys. JETP* 38 (2), 291–297, translation from Russian of A. V. Gurevich and L. P. Pitaevskii, Zh. Eksp. Teor. Fiz. 65, 590–604 (August 1973).

HAYES, B. & SHEARER, M. 1999 Undercompressive shocks and Riemann problems for scalar conservation laws with non-convex fluxes. *Proc. R. Soc. Edin.* A 129 (4), 733–754.

HELFRICH, K.R. & MELVILLE, W.K. 2006 Long nonlinear internal waves. *Annu. Rev. Fluid Mech.* 38, 395–425.

HENYEY, F.S. & HOERING, A. 1997 Energetics of borelike internal waves. *J. Geophys. Res.* 102 (C2), 3323–3330.

HOEFER, M.A., SMYTH, N.F. & SPRENGER, P. 2019 Modulation theory solution for nonlinearly resonant, fifth-order Korteweg-de Vries, nonclassical, traveling dispersive shock waves. *Stud. Appl. Maths* 142 (3), 219–240.

HOLLOWAY, P., PELINOVSKY, E. & TALIPOVA, T. 2001 Internal tide transformation and oceanic internal solitary waves. In *Environmental Stratified Flows* (ed. R.H.J. Grimshaw), pp. 29–60. Kluwer.

HOLLOWAY, P.E., PELINOVSKY, E. & TALIPOVA, T. 1999 A generalized Korteweg-de Vries model of internal tide transformation in the coastal zone. *J. Geophys. Res.* 104 (C8), 18333–18350.

HORN, D.A., IMBERGER, J., IVEY, G.N. & REDEKOPP, L.G. 2002 A weakly nonlinear model of long internal waves in closed basins. *J. Fluid Mech.* 467, 269–287.

IACOCCA, E., SILVA, T.J. & HOEFER, M.A. 2017 Breaking of Galilean invariance in the hydrodynamic formulation of ferromagnetic thin films. *Phys. Rev. Lett.* 118 (1), 017203.

IL'ICHEV, A.T. & TOMASHPOLSKII, V.Y. 2015 Soliton-like structures on a liquid surface under an ice cover. *Theor. Math. Phys.* 182 (2), 231–245.

IVANOV, S.K. 2020 Riemann problem for the light pulses in optical fibers for the generalized Chen-Lee-Liu

LEACH, J.A. 2012 An initial-value problem for the defocusing modified Korteweg–de Vries equation. *J. Differ. Equ.* 252 (2), 1032–1051.

LEACH, J.A. 2013 An initial-value problem for the modified Korteweg-de Vries equation. *IMA J. Appl. Math.* 78 (6), 1196–1213.

LEFLOCH, P.G. 2002 *Hyperbolic Systems of Conservation Laws*. Birkhauser.

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