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Statistical properties of avalanches in networks

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We characterize the distributions of size and duration of avalanches propagating in complex networks. By an avalanche we mean the sequence of events initiated by the externally stimulated excitation of a network node, which may, with some probability, then stimulate subsequent excitations of the nodes to which it is connected, resulting in a cascade of excitations. This type of process is relevant to a wide variety of situations, including neuroscience, cascading failures on electrical power grids, and epidemiology. We find that the statistics

that our analysis allows us to identify how changes in network structure affect the parameters of the statistical distributions of avalanche size and duration. Moreover, our theory allows us to obtain the statistics of avalanches starting at particular network nodes.

This paper is organized as follows. In Sec. II we describe our model for avalanche propagation in networks. In Secs. III and IV we analyze the statistics of avalanche duration and size. In Sec. V we validate our analysis through numerical experiments. Section VI presents further discussion and conclusions.

II. FORMULATION

To model the propagation of avalanches in a network, we consider a network of N nodes labeled $= 1,2,\ldots,N$. Each node has a state ˜

By definition (see also Appendix A), (t) is a bounded, increasing function of t and therefore it must converge to a value $\lim_{t\to\infty} (t) = \leqslant 1$, which can be interpreted as the probability that an avalanche starting at node has finite duration. Our analysis will be based on whether or not this limit is strictly less than one or equal to one. As shown in Appendix A, this is determined by the Perron-Frobenius eigenvalue of A, : If $\leqslant 1$, then $\lim_{t\to\infty}$

the convergence of (t) to 0 is slower than exponential, we look for a solution

$$E[\;^{-}\quad|W\quad] = \quad(\;),$$
$$E[\;^{-}\quad|Z\quad] = 1,$$

substitution into Eq. (28) gives

$$E[\;^{-}\quad|Z\quad \square\, W\quad] = \frac{(1-A\quad)+\quad A\qquad(\;)}{(1-A\quad)+\quad A}. \tag{29}$$

Inserting this into Eq. (27), we obtain one of our main results,

$$(\;) = \;^{-}\;\prod_{=1}^{N}\frac{(1-A\quad)+\quad A\qquad(\;)}{(1-A\quad)+\quad A}. \tag{30}$$

Defining $(\;) = \quad(\;) - 1$ and the matrix H with entries

$$H\quad =$$

V. NUMERICAL EXPERIMENTS

In this section we test the theoretical predictions of the preceding sections by directly simulating the process described in Sec. II on computer-generated networks. We first describe the processes used to construct networks and simulate avalanches.

Networks were constructed in two steps. First, binary networks (with adjacency matrix entries $\hat{A}$ □ $\{0,1\}$) were constructed via an implementation of the configuration model [30], using $N = 10^5$ nodes, with nodal degrees drawn from a power-law distribution with exponent 3.5, i.e., the probability that a node has degree is proportional to $^{-3.5}$. Second, each nonzero entry $\hat{A}$ was given a weight, drawn from a uniform distribution $\mathcal{U}[0,1]$. We then calculated the Perron-Frobenius eigenvalue of this weighted matrix, $\hat{}$, and multiplied the matrix by $/\hat{}$, resulting in a matrix A with the desired eigenvalue . We simulated avalanches for networks with between 0.5 and 1.5, sampling more finely for values close to 1.

Each simulated avalanche was created by first exciting a single network node, chosen uniformly at random, and then calculating the size and duration of the resulting avalanche as defined in Eqs. (1) and (2). If the resulting avalanche lasted for more than 10^6 time steps, we considered it as having infinite duration and infinite size. In all cases the initial

start at the nodes that tend to generate the largest avalanches. As shown in Fig. 7, the naive prediction that the nodes with the largest out-degree generate the largest avalanches is not necessarily true when the networks have nontrivial structure, such as degree correlations.

In developing our theory we made some assumptions that we now discuss. First, we assumed that the network was locally treelike. This allowed us to treat avalanches propagating to the neighbors of a given node as independent of each other. While this is a good approximation for the networks we used, it is certainly not true in general. In particular, avalanches propagating separately from a given node might excite the same node as they grow. The result is that the number of nodes that the avalanches excite in the simulation may be less than what the theory would predict. In running our simulations we addressed this issue in two ways. First, we kept track of the number of times two branches of the same avalanche simultaneously excited the same node , finding it to be an increasing function of avalanche size and Perron-Frobenius eigenvalue, yet still negligible when compared to the total number of excitations. In addition, each time such an event occurred, we separately generated an avalanche starting from the doubly excited node and corrected both the size and duration of the original avalanche by incorporating these additional avalanches. We found that doing this had no appreciable effect on the measured distributions and so all figures shown in this paper are produced from simulation data without the additional compensating avalanches included. This and the fact that the numerical simulations are described well by the theory suggest that the interaction of avalanches propagating to different neighbor nodes can be safely neglected in the networks studied. The performance of our theory in networks that are not locally treelike, such as networks with a high degree of clustering, is left for future research. Another approximation we used is that the Perron-Frobenius eigenvalue is well separated from the rest of the spectrum. This is a good approximation in networks without well-defined communities, but can break down in networks with strong community structure [27].

Finally, we note that our results show that the experimental signatures of criticality in neural systems (characterized by a power-law distribution of avalanche size and duration with exponents $-3/2$ and -2, respectively [4,5,11,12]) are robust

APPENDIX B: $\lambda > 1 \Rightarrow \lambda_D < 1$

In this appendix we argue that the Perron-Frobenius eigenvalue of the similar matrices H and D is less than one when the Perron-Frobenius eigenvalue of A is greater than one:

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