Approximating the largest eigenvalue of network adjacency matrices

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The largest eigenvalue of the adjacency matrix of a network plays an important role in several network processes (e.g., synchronization of oscillators, percolation on directed networks, and linear stability of equilibria of network coupled systems). In this paper we develop approximations to the largest eigenvalue of adjacency matrices and discuss the relationships between these approximations. Numerical experiments on simulated networks are used to test our results.

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I. INTRODUCTION

In recent years, there has been much interest in the study of the structure of networks arising from real world systems [1]. Another concern has been dynamical processes taking place on networks, and the impact of network structure on such dynamics. The largest eigenvalue of the network adjacency matrix has emerged as a key quantity important for the study of a variety of different dynamical network processes. For example, large ensembles of heterogeneous dynamical systems can undergo a transition to synchronization as the coupling strength k between the systems is increased. For a large class of networks and dynamical systems, the value of k at which the transition to synchronization takes place is given by $k_c = k_0/\Lambda$, where k_0 depends only on the dynamics of the uncoupled dynamical systems and Λ is the largest eigenvalue of the network adjacency matrix [2]. The largest eigenvalue Λ is also important in percolation on directed networks [3], linear stability of the fixed points of systems of network-coupled ordinary differential equations [4], and several other examples in physics and chemistry [5,6]. In this paper we study methods of obtaining approximations to Λ for the case of large complex networks.

We consider a network as a directed graph with N nodes, and we associate to it an $N \times N$ adjacency matrix whose elements A_{ij} are one if there is a directed edge from i to j and zero otherwise. (We require no self-edges $A_{ii}=0$ but allow bidirectional edges $A_{ij}=A_{ji}=1$.) We denote the largest eigenvalue of A by Λ (assuming that the graph is connected, the eigenvalue of A with the largest magnitude is unique, real, and positive by the Perron-Frobenius theorem [6]). Furthermore, we note that it is often the case that the largest eigenvalue is well separated from the second largest eigenvalue (see Fig. 1).

The properties of Λ have been studied in the context of small or regular graphs [5] and in classical Erdös-Renyi random graphs [7]. However, the structure of real world networks is usually more complex, as demonstrated by the fact that the degree distribution in a large number of examples has been found to be highly heterogeneous (often following a power law [8]), where the out-degree and in-degree of a node i are defined by $d_i^{\text{out}} = \sum_{j=1}^{N} A_{ij}$ and $d_i^{\text{in}} = \sum_{j=1}^{N} A_{ji}$.

many network realizations). Specifically, they found that

$$\lambda \approx \begin{cases} \hat{\beta} & \hat{\beta} \ll \bar{d}_{\max} \ln N, \\ \sqrt{\bar{d}_{\max}} & \sqrt{\bar{d}_{\max}} \ll \hat{\beta} \ln^2 N. \end{cases} \tag{2}$$

Some previous results for dynamical processes in networks have been stated in terms of the quantity $\hat{\beta}$, for example, the synchronization threshold in the mean-field theory of coupled oscillators in networks [2,16,17] and the network percolation and epidemic spreading thresholds [18,19].

Real world networks often have some amount of edge degree correlations [20], i.e., a node of a given degree is more likely to be connected to nodes with certain other degrees than would be expected on the basis of chance. Networks in which high degree nodes connect preferentially to high (low) degree nodes, and vice versa, are called assortative (disassortative). Such correlations can affect dynamical processes on networks, as has been demonstrated, for example, in epidemic spreading models and percolation [21–23].

We also emphasize that the in- and out-degrees at a node can have different distributions [i.e., $P_{\text{in}}(d^{\text{in}}) \neq P_{\text{out}}(d^{\text{out}})$], as has been noted for some corporate information and genetic networks [24,25], and that there are potential correlations between the in and out degrees at the same node, which can also significantly affect the largest eigenvalue. We call these correlations node degree correlations.

The rest of this paper is organized as follows. Section II reviews the characterization of degree correlations. Section III develops the theory of the maximum eigenvalue λ for the case of networks that satisfy a certain Markovian property. Some of the considerations of Sec. III are similar to theory in previous papers, where, however, those previous consider-

and out edges does not necessarily make a subnetwork with large eigenvalue. Regarding the latter point, consider Fig. 2(b) as compared to Fig. 2(a). This directed network has all its eigenvalues zero, because no pair of in and out edges connects the same two nodes.

B. Edge degree correlations

For our subsequent analysis it is useful to introduce an edge degree correlation coefficient Π characterizing the correlation between the in-degree at node i and the out-degree at node j, where a directed edge goes from i to j,

$$\Pi \equiv \langle d_i^{\text{in}} d_j^{\text{out}} \rangle_e / \langle d_i^{\text{in}} \rangle_e \langle d_j^{\text{out}} \rangle_e, \tag{9}$$

with $\langle Q_{ij} \rangle_e$ denoting an average over edges

$$\langle Q_{ij} \rangle_e \equiv \sum_{i,j} A_{ij} Q_{ij} / \sum_{i,j} A_{ij}. \tag{10}$$

Using Eq. (0

$$\square_C \square_z = C \square_z, \qquad (19$$

$= \lim_{k \to \square} n_{k+1}$

network with power law degree distribution generated as described above with $N=25\,000$, $\langle d\rangle=100$, and exponent $\gamma=2.5$. In Fig. 4(b) we plot the same quantities for a directed network with $N=10\,000$, $\langle d\rangle=20$, and exponent $\gamma=2.5$. When constructing the directed network, we chose the target $\hat{d}_k^{\text{in}}$ independently from $\hat{d}_k^{\text{out}}$ so that there are no node degree correlations ($\sigma=1$). In these plots there is no discernible difference between the approximation λ_C and the actual values of λ. We observe that the largest eigenvalue depends strongly on the correlation coefficient: in the undirected case, it increases more than three times as σ varies from 0.4 to 1.7. Also, we see that in these examples the linear approximation works for $|\sigma-1|\lesssim 0.2$, but fails for larger values of $|\sigma-1|$. In the undirected case, λ is larger than the linear approximation, which follows from Eq. (6) if we set $q_i=d_i$. We also note that in the undirected network there are strong node degree correlations ($\sigma=\hat{\lambda}/\langle d\rangle\sim 2.5$), but this does not affect the quality of the approximations.

In Fig. 5 we show the eigenvector u_i for the network of Fig. 4(b) at $\sigma\approx 0.9$ plotted against the corresponding approximation $(u_C)_i$, Eq. (20), using an arbitrary scale. Up to a normalization factor, there is good agreement between the true value and its Markov estimate $[(u_C)_i\approx u_i]$.

V. CONCLUSION

In this paper we have considered several approximations to the largest eigenvalue of the adjacency matrix of large, directed networks. The mean-field result (3) appears to apply well to networks whose neighboring nodes are uncorrelated in their degrees. The linear approximation (34) applies for sufficiently small correlation, while the Markov model (18) applies for arbitrarily strong degree correlations between neighbors. The price to be paid for a more refined approximation is the requirement of greater knowledge of the network [e.g., use of Eq. (18) requires knowledge of $P(z'|z)$ which is not required for the two other less refined approximations].

We caution that, although we have obtained good agreement between the theory and numerical results on simulated networks, this may not necessarily carry through for real networks encountered in practice. In particular, the Markov assumption of Eq. (17) may not always hold (e.g., due to community structure [26], clustering, or edge degree correlations extending over more than one edge between nodes). This remains a topic for further study.

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