Department of Applied Mathematics

Preliminary Examination in Numerical Analysis

August 19, 2019 , 10 am – 1 pm.

Submit solutions to four (and no more) of the following six problems. Show all your work, and justify all your answers. Start each problem on a new page, and write on one side only. No calculators allowed.
Do not write your name on your exam. Instead, write your student number on each page.

Problem 1. Root finding

Consider a random

(a) Suppose you apply the equispaced composite trapezoid rule with n subintervals to approximate ${}_0^L f(x)dx$. What is the asymptotic error formula for the error in the limit n with L fixed?

(b) Suppose you consider the quadrature from (a) to be an approximation to the full integral from 0 to . How should L increase with n to optimize the asymptotic rate of total error decay? What is the rate of error decrease with this choice of L?

(c) Make the following change of variable

The asymptotic error formula is [illegible]

[illegible] have a singularity at $y = 1$ [illegible]

$$[illegible] \, (1 - y_g^2)^2 (1 + y_g)^{[illegible]}$$

From this expression it's clear that to avoid infinite $F'(1)$ we need $\alpha > 2$. If $0 < \alpha < 2$ then we [illegible]

[illegible] has a faster rate of convergence; otherwise [illegible] to use the domain tr[illegible]

Problem 3. Linear algebra

(a) Given two self-adjoint (Hermitian) matrices, A and B, where B is a positive (or negative) definite matrix, show that the spectrum of the product of such matrices, AB, is real.

(b) Using $2 \quad 2$ matrices, construct an example where the product of two real symmetric matrices does not have real eigenvalues.

Solution:

(a)

Consider the eigenvalue problem $ABx \quad x, x \quad 0$. We have $\quad ABx, Bx \qquad x, Bx \quad$ and observe that $\quad ABx, Bx \quad$ is real since for any y, $\quad Ay, y \qquad y, Ay \qquad \overline{Ay, y}$. Also for $x \quad 0, \quad x, Bx \qquad Bx, x \qquad 0$ since B is a positive self-adjoint operator (less than zero if B is negative definite). We therefore conclude that is real.

(a) Alternative solution

Say B is positive definite (PD) (else use same argument as below with -B). $B^{1/2}$ then exists and is also PD (form it with same eigenvectors as for B but use square root for each eigenvalue). AB has the same eigenvalues as $B^{1/2}(AB)B^{\;1/2} \quad B^{1/2}AB^{1/2}$ (similarity transform). The latter matrix is Hermitian, so its eigenvalues are all real.

(b)

Consider an example with

$$A \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

where the eigen[illegible] of both matrices are 1 and [illegible]

Problem 4. **Interpolation / Approximation**

Let function $f \in C^{n+1}[a,b]$, $|f^{(n+1)}(x)| \le M$ and $E_n(f)$ be the error of its best approximation by a polynomial of degree *n*. Show that the accuracy of the best polynomial approximation improves rapidly as the size of the interval $[a,b]$ shrinks, i.e., show that

$$E_n(f) \le \frac{2M}{(n+1)!}\left(\frac{b-a}{4}\right)^{n+1}.$$

Hint: Use the Chebyshev nodes $x_l = \tfrac{1}{2}(b+a) + \tfrac{1}{2}(b-a)\cos\pi\tfrac{2l+1}{2n+2}$ to construct a polynomial approximation of *f*.

Solution. The error term of interpolation by a polynomial of degree $n+1$ is

$$x = x(y) = \frac{1}{2}(b+a) + \frac{1}{2}(b-a)\,y, \quad y \in [-1,1]$$

Alternative (similar) solution:

Consider first $[a,b] = [-1,1]$. The formula for the error in Lagrange interpolation gives $|E_n(f)| \le \max_x \frac{|f^{(n+1)}(x)|}{(n+1)!}\left|\prod_{l=0}^{n}(x-x_l)\right|$. With Chebyshev nodes, $\left|\prod_{l=0}^{n}(x-x_l)\right| = \left|\frac{1}{2^n}T_{n+1}(x)\right| \le \frac{1}{2^n}$. Stretching / contracting / shifting the interval from one of length $(b-a)$ to one of length 2 does not affect function vales, it multiplies first derivatives by $\frac{b-a}{2}$, second derivatives by $\left(\frac{b-a}{2}\right)^2$, ... , n+1st derivative by $\left(\frac{b-a}{2}\right)^{n+1}$. For the original interval $[a,b]$, we thus get $|E_n(f)| \le \frac{1}{(n+1)!}\frac{1}{2^n}M\left(\frac{b-a}{2}\right)^{n+1} = \frac{2M}{(n+1)!}\left(\frac{b-a}{4}\right)^{n+1}$.

Problem 5. Numerical ODE

There exists a one parameter family of 2-stage, second order Runge Kutta methods for solving the ODE $y' \quad f(x, y(x))$. With step size h in the x-direction, and the parameter arbitrary, these can be written as

$$d^{(1)} \quad h\,f(x_n, y_n)$$
$$d^{(2)} \quad h\,f(x_n \quad h, y_n \quad d^{(1)})$$

Problem 6. Numerical PDE

(a) Verify that the PDE — $\frac{3}{3}$