A draft of INRIA lectures, May 1991

Wavelets, Multiresolution Analysis and Fast Numerical Algorithms

Bey n

M po e o eq e o de N ope on o co p e e

$$p_j \; - \sum_{i \neq j} \frac{q_i q_j}{| \quad i}$$

e e e od y e e ed de ce fo ed c n p d . en eq on o
p e ne y e fo e co of n n e en y cond on n e of e e n
ce f n e d of n e d . ence o n e e e en ep e en on e e e ep
e en on of e de e n e e e en p e od c on

e n y e

II.1 Multiresolution analysis.

e de n on of e e o on n y no on n od ced y Meye , nd M , c p e e en

f ee n en e of e en o o of ene y e —
o e ne e n e d of e e en e

$$\mathbf{V_n} \subset \quad \subset \mathbf{V} \subset \mathbf{V} \subset \mathbf{V} \quad \mathbf{V} \subset \mathbf{L} \; \mathbf{R^d}$$

n n e c e z on e p ce $\mathbf{V}$ n e d en on

II.2 The Haar basis

o on e e on y e p e of e e o on n y y n De n on Cond on e [redacted] Co p n n [redacted] o p e of e o e de e n e ec e nd p o de f p o o ype fo n e c e pe en on f d — en [redacted] $\mathbf{j;k}$ — $\mathbf{j=}$ $\mathbf{j}$ — $\in \mathbf{Z}$ fo ed y e d on nd n on of n e f nc on

$$\begin{cases} & \text{fo} \\ — \quad - & \text{fo} \quad \leq \\ & \text{e e e} \end{cases} \tag{7}$$

n c — e e e c c e c f nc on of e n e o e c $\mathbf{j;k}$ — $\mathbf{j=}$ $\mathbf{j}$ — $\in \mathbf{Z}$ e of $\mathbf{V_j}$ nd $\mathbf{j;k}$ — $\mathbf{j=}$ $\mathbf{j}$ — $\in \mathbf{Z}$ e of $\mathbf{W_j}$

e deco po on of f nc on n o [redacted] n o de N p oced e en N — $\mathbf{n}$ p e of f nc on c y fo p c y e o of e of ed e e of f on n e of en $\mathbf{n}$

$$\mathbf{k} — \mathbf{n=} \int_{n\mathbf{k}}^{n\;\mathbf{k}+\;)} f \quad d$$

e o n [redacted] coe c en

$$d_{\mathbf{k}}^{\mathbf{j}+} — \frac{}{\sqrt{}} \;\mathbf{^j_k} - \mathbf{^j_k}$$

nd e e

$$\mathbf{^{j+}_k} — \frac{}{\sqrt{}} \;\mathbf{^j_k} \quad \mathbf{^j_k}$$

fo — n— nd — $\mathbf{n}$ $\mathbf{j}$ — e y o ee e n e o e of coe c en $d_{\mathbf{k}}^{\mathbf{j}}$ $\mathbf{^j_k}$ n eq e N— dd on nd N p c on n o d en on e e e o n — 96.1 f 5.64 -14.4 Td f 5.7564 0 Td 1f d d en o e nd d fo e econd de ned y e e of ee nd of f nc on ppo ed on e $_{j;k}$ $_{j;k'}$ y $_{j;k}$ $_{j;k'}$ y nd $_{j;k}$ $_{j;k'}$ y e e e c c e c f nc on of e n e nd $_{j;k}$ $:=$ $_{j=}$ $_{j}$ $-$ ep e en n n ope o n e d o e non nd d fo e e no o y eco e c e e

By con de n n n e ope o

$$f \; := \int \; y \; f \; y \; dy$$

nd e p nd n e ne n o d en on e nd fo C de on Zy nd nd p e do d en ope o e dec y of en e f nc on of e d nce f o e d on f e n e e ep e en on n n e o n e ne e e c of ope o e en y n e o d on e ne e oo y f o e d on o e p e e ne y of C de on Zy nd ope o fy e e e

$$| \quad y | \leq \frac{}{| \; - y|}$$

$$| \,_{x}^{M} \quad y | \quad | \,_{y}^{M} \quad y | \leq \frac{C_M}{| \; - y|^{\; + M}}$$

fo o e $M \geq$ Le $M :=$ n nd con de

$$^{j}_{kk'} := \int \int \; y \; _{j;k} \; _{j;k'} \; y \; d \; dy$$

e e e e e d nce e een $| \; - \; '| \geq$ nce

$$\int \; _{j;k} \; d := $$

e e

$| \quad ; \quad y \quad f \; r \quad x$

e on e n n e dec y n c en o e co p n n p c c o e f e dec y nece y o e f nc on e e n n o en e n n o en e e on e fo n n p c c o e con o n e con n n e co p e y e e of e f o

II.3 Orthonormal bases of compactly supported wavelets

e q e on of e e ence of e on n y oo f nc on of Cond on e no e ed n e con c on of e o ono e e ene z n f nc on y o e nd Meye e con de on y co p c y ppo ed e e n n o en con c ed y D ec e fo o n e o of Y Meye nd M o o of e e e n d fo o e c o ce of e e e e f e e c o c o ce of e e y on y con de n fo d en on $d \geq$ e con c ed f o o e fo d

Le con de e e on n y fo L R^7 e n f d d e econd e o o on y of { − }$_{k2Z}$ p e

$$k \; - \int_{1}^{+1} - \quad d \; - \int_{1}^{+1} | \quad | \; e^{ik} \; d$$

nd e efo e

$$k \; - \int \sum_{l2Z} | \quad | \; e^{ik} \; d$$

nd

$$\sum_{l2Z} | \quad | \; -$$

n e o n

$$\sum_{l2Z}$$

no d c o o ee e . , . , . . , on e c ed e ; $\in$ **Z** e e e $\{$ $_{j;k}$ — $^{j=}$ ^{j} — $\}_{k2Z}$ fo n o ono of W_j e fo o n e D ec e , c c e ze ono e c po yno o on of c co e pond o e o ono e of co p c y ppo ed e e n n o en

Lemma II.1 Any trigonometric polynomial solution of (2.26) is of the form

$$ - \tfrac{1}{2} \quad e^{i} \quad {}_{M} \quad e^{i} $$

where $M \geq$ **is the number of vanishing moments, and where is a polynomial, such that**

$$ | \quad e^{i} \quad | \quad -P \quad \sin \tfrac{1}{2} \quad \sin^{M} \tfrac{1}{2} \quad \tfrac{1}{2} \cos $$

where

$$ P\, y \quad - \sum_{k}^{M} \quad M - \quad ! \quad y^{k} $$

and is an odd polynomial, such that

$$ \leq P\, y \quad y^{M} \quad \tfrac{1}{2} - y \quad \textbf{for} \quad \leq y \leq $$

and

$$ \sup_{y} \left[P\, y \quad y^{M} \quad \tfrac{1}{2} - y \right] \quad {}^{M} \quad) $$

e p oof of and

e e $^{j}_{k}$ nd d^{j}_{k} y e e ed pe od c eq ence e pe od n j Co p n
nd ed y e py d e e

$$\{\ _{k}\} \longrightarrow \{\ _{k}\} \longrightarrow \{\ _{k}\} \longrightarrow \{\ _{k}\} \quad \cdots$$

$$\searrow \qquad \searrow \qquad \searrow$$

$$\{d_{k}\} \qquad \{d$$

e en de ne $f_{\mathbf{m}} := f_{\mathbf{m}} - \mathbf{m} \cdot f$ e e $\mathbf{m}$ c o en o $\langle f_{\mathbf{m}}\ \mathbf{M} \rangle :=$ fo
$:=$ M c e n e de ed o o on y o $\mathbf{M}$ y e con n e o

e e $\mathbf{V}_j^{M}$ en n e nd e p ce $\mathbf{W}_j^{M;}$ e o on co p e en of $\mathbf{V}_j^{M;}$ n $\mathbf{V}_j^{M;}$

$$\mathbf{V}_j^{M;} = \mathbf{V}_j^{M;} \quad M \quad \mathbf{W}_j^{M;}$$

e p ce $\mathbf{W}^{M;}$ p nned y e o ono

$$\{ \quad i \quad l\, y \quad i \quad l\, y \quad i \quad l\, y \quad = \quad M\}$$

e e $\{ \; m\}_{m}^{m \quad M}$ n o ono fo $\mathbf{V}^{M}$ c e e en on e M e e en n n o en e o on o e po yno $^{i}y^{l}$ = $M -$

e p ce $\mathbf{W}_j^{M;}$ p nned y d on nd n on of e f nc on of $\mathbf{W}^{M;}$ nd e of $\mathbf{L}$, con of e f nc on nd e o o de po yno $^{i}y^{l}$ = $M -$

e no e e o d en on e e eq e M d en co n on of one d en on f nc on e e M e n e of n n o en On e o e nd e o d en on o ned y n co p c y ppo ed e e , eq e on y ee c co n on c p e e con c on of e non nd d fo e ec on

II.5 A remark on computing in the wavelet bases

n y e no e once e e een c o en co p e e y de e ne e f nc on nd nd e efo e e e on n y n n e e n o on n p ope y con c ed o e f nc on nd e ne e co p ed D e o e ec e de n on of e e e e n p on e pe fo ed e q d e o e nd e en f ey n o e q n e c ed nd A n e p e e co p e e o en of e n f nc on e e p e on fo e o en

$$\mathcal{M}_1^{m} = \int \quad {}^{m} \quad d \quad = \quad M -$$

n e of e e coe c en $\{ \; k\}_{k}^{k \quad L}$ y e fo nd n fo fo

$$= \quad \prod_{j} \quad j$$

e e

$$= \sum_{k} {}^{k} \; {}_{k} e^{ik}$$

e o en $\mathcal{M}_1^m$ e o ned n e c y n e de ed cc cy y ec e y ene n eq ence of ec o $\{\mathcal{M}_r^m\}_m^M$ fo r =

$$\mathcal{M}_{r+}^m = \sum_j^m \quad ! \quad {}^{jr}\mathcal{M}_r^m \; {}^j\mathcal{M}^j$$

n

$$\mathcal{M}^m = {}^m \tfrac{1}{2} \sum_k^k {}^k \; {}^m \quad = \quad M -$$

c ec o $\{\mathcal{M}_r^m\}_m^M$ ep e en M o en of e p od c n r e nd e on con e e p d y No ce e ne e co p ed e f nc on e f

e non nd d nd nd d fo

III.1 The Non-Standard Form

Le e n ope o

$$\mathbf{L} \quad \mathbf{R} \to \mathbf{L} \quad \mathbf{R}$$

e e ne y De n n p o ec on ope o on e p ce $\mathbf{V}_j$; $\in \mathbf{Z}$

$$P_j \quad \mathbf{L} \quad \mathbf{R} \to \mathbf{V}_j$$

$$P_j f \quad = \sum_k \langle f \quad _{j;k} \rangle \quad _{j;k}$$

nd e p nd n n e e op c e e e o n

$$= \sum_{j \in \mathbf{Z}} \quad _j \quad _j \quad _j \quad P_j \quad P_j \quad _j$$

e e

$$_j = P_j \quad - P_j$$

e p o ec on ope o on e p ce $\mathbf{W}_j$ f e e e co e n en n e d of e e e

$$= \sum_{j \quad 1} \quad _j \quad _j \quad _j \quad P_j \quad P_j \quad _j \quad P_n \quad P_n$$

nd f e e; = e ne e en

$$= \sum_{j} \quad _j \quad _j \quad _j \quad P_j \quad P_j \quad _j \quad P_n \quad P_n$$

e e $\sim$ $= P \quad P$ d e z on of e ope o on e ne e p n on e nd deco po e ope o n o of con on f o d en e e

e non nd d fo ep e en on e of e ope o c n of p e

$$= \{A_j \quad B_j \quad _j\}_{j \in \mathbf{Z}}$$

c n on e p ce $\mathbf{V}_j$ nd $\mathbf{W}_j$

$$A_j \quad \mathbf{W}_j \to \mathbf{W}_j$$

$$B_j \quad \mathbf{V}_j \to \mathbf{W}_j$$

$$\mathbf{W_j} \to \mathbf{V_j}$$

e e e ope o $\{A_j\ B_j\ \ j\}_{j2Z}$ e de ned A_j := j j B_j := j P_j nd j := P_j j

e ope o $\{A_j\ B_j\ \ j\}_{j2Z}$ d ec e de n on e e on

$$\text{j} := \begin{pmatrix} A_{j+} & B_{j+} \\ j+ & j+ \end{pmatrix}$$

e e ope o j := P_j P_j

$$\text{j} \ \mathbf{V_j} \to \mathbf{V_j}$$

nd e ope o ep e en ed y e × n pp n

$$\begin{pmatrix} A_{j+} & B_{j+} \\ j+ & j+ \end{pmatrix} \ \mathbf{W_{j+}} \oplus \mathbf{V_{j+}} \to \mathbf{W_{j+}} \oplus \mathbf{V_{j+}} \qquad (4)$$

f e e co e n en

$$:= \{\{A_j\ B_j\ \ j\}_{j2Z\ j\ n\ n}\}$$

e e n := P_n P_n f e n e of e n e en := n n nd e ope o e o nzed oc of e ee e nd

Le e e fo o n o e on

e ope o A_j de e e n e c on on e e on y nce e p ce $\mathbf{W_j}$ n n e e en of e d ec n

e ope o B_j j n nd de e e n e c on e een e e nd co e nd eed e p ce $\mathbf{V_j}$ con n e p ce $\mathbf{V_{j^0}}$ ^{0} ee

e ope o j n e ed e on of e ope o j

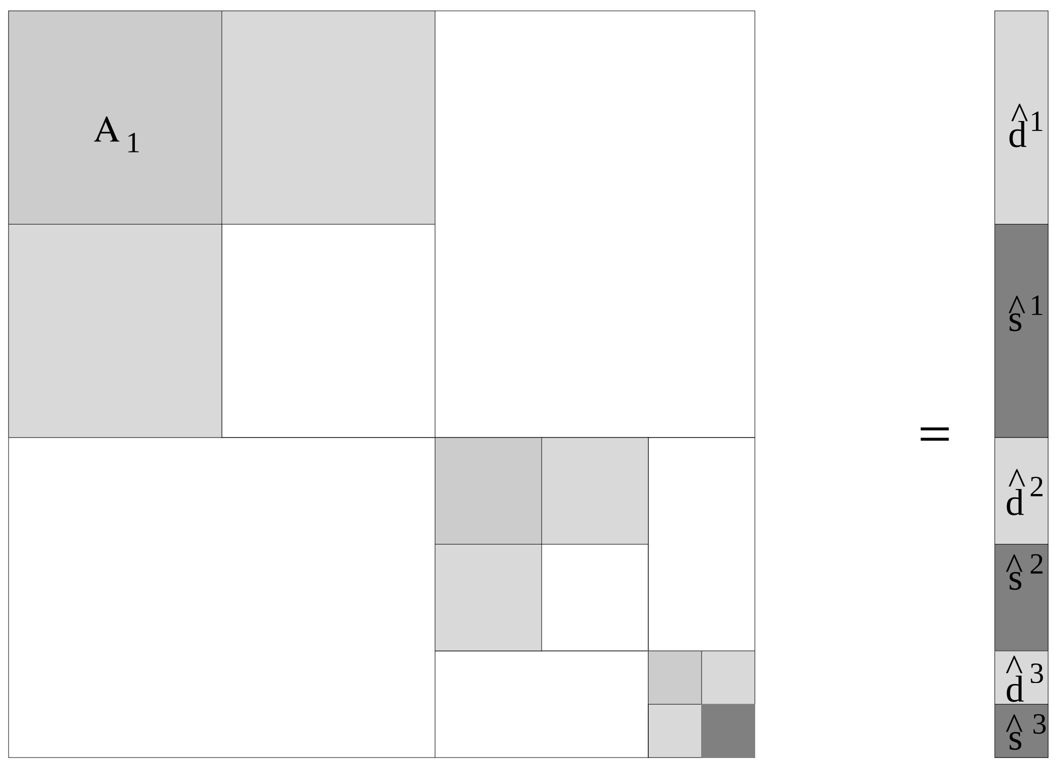
e ope o A_j B_j nd j e ep e en ed y e ce j j nd j

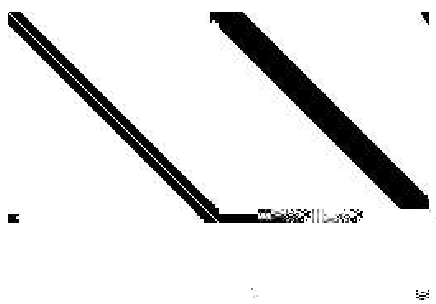
$$\text{j}_{k;k^0} := \int\int y \ \ \text{j;k} \ \ \text{j;k}^0\ y \ d\ dy$$

$$\text{j}_{k;k^0} := \int\int y \ \ \text{j;k} \ \ \text{j;k}^0\ y \ d\ dy \qquad (7)$$

nd

$$\text{j}_{k;k^0} := \int\int y \ \ \text{j;k} \ \ \text{j;k}^0\ y \ d\ dy$$


A_1
=
$\hat{d}^1$
$\hat{s}^1$
$\hat{d}^2$
$\hat{s}^2$
$\hat{d}^3$
$\hat{s}^3$



e An e p e of n e non nd d fo ee p e

e ope o j ep e en ed y e j

$$\int\int {}^{j}_{k;k'} = \quad y \quad {}_{j;k} \quad {}_{j;k'}\, y \quad d \quad dy$$

en e of coe c en ${}_{k;k'}$ $' =$ $N =$ epe ed pp c on of e fo p od ce

$${}^{j}_{i;l} = \sum_{k;m}^{L} \quad k \quad m \quad {}^{j}_{k+\ \ i+\ \ ;m+\ \ l+}$$

III.2 The Standard Form

e nd d fo o ned y ep e en n

$$\mathbf{V_j} = \bigoplus_{j'>j} \mathbf{W_{j'}}$$

nd con de n fo e c e e ope o $\{B_j^{j'}, \Gamma_j^{j'}\}_{j'>j}$

$$B_j^{j'} : \mathbf{W_{j'}} \to \mathbf{W_j}$$

$$\Gamma_j^{j'} : \mathbf{W_j} \to \mathbf{W_{j'}}$$

f e e e co e en n en n e d of e e

$$\mathbf{V_j} = \mathbf{V_n} \bigoplus_{j'=j+1}^{n} \mathbf{W_{j'}}$$

n c e e ope o $\{B_j^{j'}, \Gamma_j^{j'}\}$ fo $j' =$ n e e e n nd n dd on fo e c e e e ope o $\{B_j^{n+1}\}$ nd $\{\Gamma_j^{n+1}\}$

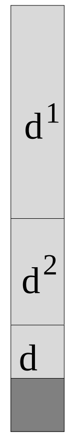
$$B_j^{n+1} : \mathbf{V_n} \to \mathbf{W_j}$$

$$\Gamma_j^{n+1} : \mathbf{W_j} \to \mathbf{V_n}$$

n no on $\Gamma_n^{n+1} = \Gamma_n$ nd $B_n^{n+1} = B_n$ f e n e of e n e nd **V** n e d en on en e nd d fo ep e en on of $= P$ P

$$= \{A_j, \{B_j^{j'}\}_{j'=j+1}^{n}, \{\Gamma_j^{j'}\}_{j'=j+1}^{n}, B_j^{n+1}, \Gamma_j^{n+1}, \dots n\}_{j=1,\dots,n}$$

e ope o e o n zed oc of e ee e nd e e f e ope o C de on Zy nd o p e do eratoj /R36 (3aV)Tj /R360 Td (Figu65



d1
d2
d

=

e co p e on of ope o

e co p e on of ope o o n o e o d e con c on of e p e ep e en on n o ono e d ec y ec e peed of co p on o e e co p e on of d of e fo e p e c e ed y e od o e n nd n p e ep e en on n e y e deq e fo e pp c on e co p e on of ope o c fo ep e en on n n o de o e ec e y co p e n e p e fo e nd d nd non nd d fo of ope o n e e e y e e ed co p e on e e e fo de c of nd non nd d fo of ope

the matrices ^{j}, ^{j}, ^{j} (3.16) - (3.18) of the non-standard form satisfy the estimate

$$| {}^{j}_{i;l}| \quad | {}^{j}_{i;l}| \quad | {}^{j}_{i;l}| \leq \frac{C_M}{| - |^{M+}} \qquad 7$$

for all $| - | \geq M$.

con de on pp y n e c e of p e do d en ope o Le e p e do d en ope o y o de ned y e fo

$$f \quad - \quad e^{ix} \int \quad f \quad d \quad - \int \quad y \; f \; y \; dy$$

e e e d on e ne of

Proposition IV.2 **If the wavelet basis has M vanishing moments, then for any pseudo-di erential operator with symbol of and of satisfying the standard conditions**

$$| \quad {}_{x} \quad | \leq C \; ; \quad | \; | \quad +$$

$$| \quad {}_{x} \quad | \leq C \; ; \quad | \; | \quad +$$

the matrices ^{j}, ^{j}, ^{j} (3.16) - (3.18) of the non-standard form satisfy the estimate

$$| {}^{j}_{i;l}| \quad | {}^{j}_{i;l}| \quad | {}^{j}_{i;l}| \leq \frac{{}^{j} \; C_M}{| - |^{M+}}$$

for all integer , .

f e pp o e e ope o ^{N} y e ope o $^{N;B}$ o ned f o ^{N} y e n o ze o coe c en of ce $^{j}_{i;l}$ $^{j}_{i;l}$ nd $^{j}_{i;l}$ o de of nd of d $B \geq M$ o nd e d on en e y o ee

$$|| \; ^{N;B} - \; ^{N} || \leq \frac{C}{B^M} \; \text{o} \; N$$

e e C con n de e ned y e ene n o n e c pp c on e cc cy of c c on ed nd e p e e of e o n o c e e nd d B nd o de M e o e c o en n c nne e de ed p ec on of c c on c e ed f M ed en B o e c

$$|| \; ^{N;B} - \; ^{N} || \leq \frac{C}{B^M} \; \text{o} \; N \leq$$

y $\cdot$ — y e 4

e e f c y n y n e f nc on e e nd e nd e efo e e ope o L nd L po e o dec de f C de on Zy nd ope o o nded

Theorem IV.1 (G. David, J.L. Journe) **Suppose that the operator (3.1) satis es the conditions (4.5), (4.6), and (4.16). Then a necessary and su cient condition for to be bounded on** L **is that in (4.24) and** y **in (4.25) belong to dyadic** BMO**, i.e. satisfy condition**

$$\sup_J \frac{}{|\ |} \int_J | \qquad J \quad | \, d \leq C \qquad e\ 4$$

where is a dyadic interval and

$$J \quad \cdot \frac{—}{|\ |} \int_J \quad d \qquad e\ 7$$

p n e ope o n o e of ee e e nd e n e ep e y e d o e e e e e no e e f nc on nd e e y co p ed n e p oce of con c n e non nd d fo nd nd y e ed o p o de f e e of e no of e ope o

e d e en ope o n e e e

V.1 The operator d=dx in wavelet bases

e non nd d fo of e e co on y ed ope o y e co p ed e p c y e con c e non nd d fo of e ope o d d e e e en of A_j B_j nd r of e e d d d

e e n e e oco e on coe c en of e e $= \{ _k \}^{k}_{k} L$

$$n = \sum_{i}^{L} \, {}^{n} \, i \; i+n \quad n = \quad L -$$

e y o ee e oco e on coe c en n e en nd ce e ze o

$$k = \quad = \quad L \; -$$

y e e fyed y n o co p e | nd |

$$| = - \sum_{n}^{L} \, n \, \text{co} \, n$$

$$| = - - - \sum_{k}^{L=} k \, \text{co} \; - \; - \sum_{k}^{L=} k \, \text{co}$$

e e n e en n Co n n nd o fy e o n

$$\sum_{k}^{L=} k \, \text{co} \; =$$

nd ence nd

e e en o en of e coe c en k f o n n e y

$$\sum_{k}^{k \; L=} k \; - \; {}^{m} = \quad \text{fo} \quad \leq \quad \leq M -$$ e

nce

$$- \; {}^{m} \quad | \quad = \quad \text{fo} \quad \leq \quad \leq \; M -$$

c fo o f o e Co o d en f e 7 e d e y fy o 7 7 e f d cof e n e e e 7

$$r_i = \sum_{k}^{L} \sum_{m}^{k} k \; k \; m \; r_{i+m}$$

C n n e o de of on n nd n e f c $P_k^L \; k =$ e
e

$$r_l = r_l \sum_{n}^{L} n \; r_{l \; n} \; r_{l+n} \quad \in \mathbb{Z}$$

e e $_n$ e en n n e o n e f o
n o de o o n e e fo o n e on

$$\sum_{l \; 1}^{l \; 1} m \quad - \quad = m \sum_{l}^{l \; m} - l \quad M_l' \; m \; l \; !$$

e e

$$M_l' = \int_{1}^{+1} {}^{l} d \quad =$$

e e o en of e f nc on e on fo o p y on n o e
n fo nd n Le n z e n nd = e o n
f $M \geq$ en

$$| \quad | \; | \; | \leq C \quad | \; |$$

e e nd ence e n e n e y con e en on
fo o f o Le of e e o n

$$| \quad | \leq C \quad | \; |^{M + \log_2 B}$$

e e

$$B = \sup_{2R} | \quad e^{i} \quad |$$

D e o e cond on e e o $B = M - -$ o e
e e ence of o on of e y e of eq on e nd fo o
f o e e ence of e n e n nce e n f nc on co p c
ppo e e e on y 7 7 d ee e d o e

≁

e e

$$r \;=\; \sum_{l} r_l e^{il}$$

$$r_{\text{even}} \;=\; \sum_{l} r\; l e^{il}$$ 7

nd

$$r_{\text{odd}} \;=\; \sum_{l} r\; l+ e^{i\; l+)\; =}$$

No c n

$$r_{\text{even}} \;=\; r \qquad r$$

nd

$$r_{\text{odd}} \;=\; r \quad - r$$ e

nd n e o n f o

$$r \;=\; r \qquad r \qquad \Big[\; | \; - \quad r \quad - r \Big]$$ e

n y n e e

$$r \;=\; \quad | \; r \qquad | \; r$$ e

e n n e e o n r r nd e
n q ene of e o on of e nd fo o f o e n q ene of
e ep e en on of d d en e on r_l of e nd e con de e
ope o $_j$ de ned y e coe c en on e p ce $\mathbf{V}_j$ nd pp y o c en y
oo f nc on f nce r^j_l $^j r_l$ e e e

$$_j f \;=\; \sum_{k \in \mathbb{Z}} \; ^j \sum_{l} r_l \, f_{j;k} \; l \qquad j;k \;!$$ e

e e

$$f_{j;k} \; l \;=\; j= \int_{1}^{+1} f \qquad j \; - \qquad d$$ ee

e n ee

$$f_{j;k} \; l \;=$$

d 7 7

e e — j nd | − | ≤ j n e n e nd n e
nd e o n

$$ j f \;=\; \sum_{k \in Z} \int_{-1}^{+1} f' \quad j;k \quad d \quad j;k $$

$$ \sum_{j} \sum_{k \in Z} - r_l \int_{1}^{+1} f'' \quad j;k \quad d \quad j;k \;! $$

c e → −∞ ope o j nd d d co nc de on oo f nc on n
nd d en e y o p o e 1 =d d nd ence e o on o e
nd n q e e e on fo o no f o

Remark 2 e no e e p e on nd fo l nd l l = − l y
e p ed y c n n e o de of on n nd nd n od c n
e co e on coe c en $\sum_i^L \; n \; i \; i+n$ $\sum_i^L \; n \; i \; i+n$ nd $\sum_i^L \; n \; i \; i+n$ e
e p e on fo l e pec y p e l = e r l − r l

Examples. o e e p e e D ec e e e con c ed n
e co p e e coe c en k = M e e M e n e of n n
o en nd L = M n e on e of

$$ | = - \frac{M -}{M -} \int n^{M} \; d $$

e nd y co p n R n M d

$$ | = - C_M \sum_{m}^{M} \frac{- \; m \; co \; -}{M \quad M \quad - \quad -} $$

e e

$$ C_M = \left[\frac{M -}{M - \; e^{M}} \right] $$

y co p n e nd e e

$$ m = \frac{- \; m \; C_M}{M \quad M \quad - \quad -} \quad e e = M $$

e no e y e of en o on of ne y e on co
e c en m n e on y con c on e coe c en r l e on
n e e coe c en r l e e e fo D ec e e e ed n e
of n n o en M e e e e e e e fo en

j e d e d en

j ene e
of e p d o yno e

o n eq on of opo on e p e en e e fo D ec e e e
M = e

1 M =

= − = − −

nd

r = − − − r = − −

e coe c en − − of e p e c n e fo nd n ny oo
on n e c n y c o ce of coe c en fo n e c d en on

---

2 M =

3 M = e

nd

r_4

4 M =

nd

r_4

5 M =

nd

r =

r_4 =

r =

r =

Coe c en fo M = nd M = c n e co p ed e co e pond n o p f o e fo o n e e o

Iterative algorithm for computing the coe cients r_l.

A y of o n eq on e nd e y o e n e e o e r = nd r = nd e e n e o eco p e r_l e y o e fy n e nd e ed d e o e c o ce of n z on e fo o n fo D ec e e e M = co p ed n o d p y e coe c en $\{r_l\}_l^L$ e no e r_l = $-r_l$ nd r =

V.2 The operators d^n/dx^n in the wavelet bases

o e ope o d d e non nd d fo of e ope o d^n d^n co p e e y de e ned y ep e en on on e p ce **V** e y e coe c en

$$r_l^{(n)} = \int_{-\infty}^{+\infty} \quad \frac{d^n}{d^n} \quad d \qquad l \in \mathbf{Z}$$

o e n e y

$$r_l^{(n)} = \int_{-\infty}^{+\infty} \quad {}^n |\quad| \; e^{il} \; d$$

f e ne n o e ee e e o

		Coe cients				Coe cients
	i	ι			i	ι
M = 5	1	-0.82590601185015		M = 8	1	-0.88344604609097
	2	0.22882018706694			2	0.30325935147672
	3	-5.3352571932672E-				

Proposition V.2 1. If the integrals in (5.52) or (5.53) exist, then the coe cients $r_l^{(n)}$, $l \in Z$ satisfy the following system of linear algebraic equations

$$r_l^{(n)} = \, ^n \left[r_{l} - \sum_{k}^{L=} \; _k \; r_{l \; k+}^{(n)} \; r_{l+ \; k}^{(n)} \right] \qquad (5.54)$$

and

$$\sum_{l} \, ^n r_l^{(n)} = - \, ^n n$$

where $_k$ are given in (5.19).

2. Let $M \geq n$, where M is the number of vanishing moments in (2.16). If the integrals in (5.52) or (5.53) exist, then the equations (5.54) and (5.55) have a unique solution with a nite number of non-zero coe cients $r_l^{(n)}$, namely, $r_l^{(n)} \neq$ for $-L \leq l \leq L -$. Also, for even n

$$r_l^{(n)} = r_{-l}^{(n)}$$

$$\sum_{l} \, ^n r_l^{(n)} = \quad n = \quad n - \qquad (5.57)$$

and

$$\sum_{l} r_l^{(n)} =$$

and for odd n

$$r_l^{(n)} = -r_{-l}^{(n)}$$

$$\sum_{l} \, ^n \, r_l^{(n)} = \quad n = \qquad \leq l \leq L$$

$A \quad e \; M$

e no e on e e e L = e e e e o n n
o en M = do no e e o de e ponen ee e ep e en on
of e d de e e on y f e n e of n n o en M =

e eq on fo co p n e coe c en $r_l^{(n)}$ y e e ed n e en e
p o e Le de e e eq on co e pond n o e fo d^n d ^{n} d ec y f o
e e e

$$r_l^{(n)} = \int \sum_{k \in \mathbb{Z}} | \quad | ^n \quad ^n e^{il} \, d$$

e efo e

$$r = \sum_{k \in \mathbb{Z}} | \quad | ^n \quad ^n$$

e e

$$r = \sum_l r_l^{(n)} e^{il}$$

n e e on

e
n o e nd de of nd n o e e en nd odd nd ce n
ep e y e e

$$r = {}^n \quad | r \quad | r$$

Le con de e ope o M on pe od c f nc on d f n f d

$$M f = \quad | f \quad | \quad |$$

n e e e e n e c e dence n f c of n e e nce ep
e en one of e d n e of co p n n e e e e
e e c e

N	[illegible]	[illegible]p
64	0.14545E+04	0.10792E+02
128	0.58181E+04	0.11511E+02
256	0.23272E+05	0.12091E+02
512	0.93089E+05	

e con o on ope o n e e e

n ec on e con de e co p on of e non nd d fo of con o on ope o . o con o on ope e q d e fo fo ep e en n e e ne on V e of e p e fo d e

nd e den y fo o f o
o d

nce e o en of e f nc on n eq on e e d o
po n q d e fo fo co p n e ep e en on of con o on
e ne e fo co p c y ppo ed e e fo o
e e nne fo e pec c o ce of e e e de eqn
e e e f ed o en of e f nc on n e ef p pe fo
e de

e e e n od ce d en pp o c c con n of
ne e c eq on ec o y p o c cond on pe
c y p e f e y o of e ope o o o eneo of e
c e ope o co p e e y de ned y ep e en on on **V**
Le con de o e p e of c ope o e nd
ope o of f c on d en on o n d en on

VI.1 The Hilbert Transform

e pp y o e od o e co p on of e non nd d fo e n
fo

$$\mathcal{H}f \quad y \quad p \int \frac{f}{\quad} d$$

e e p deno e p nc p e
e ep e en on of $\mathcal{H}$ on **V** de ned y e coe c en

$$r \qquad \int \quad \mathcal{H} \quad d \qquad \in \mathbf{Z}$$

c n n co p e e y de ne o e coe c en of e non nd d
$\mathcal{H} \quad \{A_j \; B_j \; \}_{j \in \mathbf{Z}} \quad A_j \quad A \quad B_j \quad B$ nd $\quad j \quad$ e e e

	Coe cients		Coe cients
l		l	
M = 6 1	-0.588303698	9	-0.035367761
2	-0.077576414	10	-0.031830988
3	-0.128743695	11	-0.028937262
4	-0.075063628	12	-0.026525823
5	-0.064168018	13	-0.024485376
6	-0.053041366	14	-0.022736420
7	-0.045470650	15	-0.021220659
8	-0.039788641	16	-0.019894368

e e e coe c en r_l of e n fo fo D ec e e e n n o en

e coe c en $r_l \in$ **Z** n fy e fo o n y e of ne e c eq on

$$r_l = r_l - \sum_{k}^{L} k \; r_{l} \; k + \; r_{l+k}$$

e e e coe c en k e en n n e nd 7 e o n e y p o c of r_l fo e

$$r_l = - O \frac{}{M}$$

By e n n e of

$$r_l = \int^{1} | \; | \; n \; d$$

e o n $r_l = -r_l$ nd r e no e e coe c en r c nno e de e ned f o eq on e nd o n e e y p o c cond on e co p e e coe c en $r_l \neq$ ny p e ed cc cy

Example.

e co p e e coe c en r_l of e n fo fo D ec e e e n n o en cc cy e coe c en fo e o ned n y p o c e no e $r_l = -r_l$ nd r

VI.2 The fractional derivatives

e e fo o n de n on of f c on de e

$$ {}_{x}f \;=\; \int_{1}^{+1} \frac{-y_{+}}{\;\;-} f\, y\, dy \qquad 7 $$

e e e con de f en 7 de ne f c on n de e e ep e en on of $_{x}$ on **V** de e ned y e coe c en

$$ r_{l} = \int_{1}^{+1} - \quad {}_{x} \quad d \qquad \in \mathbf{Z} $$

p o ded n e e

e non nd d fo $_{x} = \{A_{j}\; B_{j}\; {}_{j}\}_{j2Z}$ co p ed $A_{j} = {}^{j}A$ $B_{j} = {}^{j}B$ nd $_{j} = {}^{j}$ e e e e en $_{i\ l}$ $_{i\ l}$ nd $_{i\ l}$ of A B nd e o ned f o e coe c en r_{l}

$$ {}_{i} = \sum_{k}^{L} \sum_{k^{0}}^{L} {}_{k\ \ k^{0}}\, r_{\,i+k\ \ k^{0}} $$

$$ {}_{i} = \sum_{k}^{L} \sum_{k^{0}}^{L} {}_{k\ \ k^{0}}\, r_{\,i+k\ \ k^{0}} $$

nd

$$ {}_{i} = \sum_{k}^{L} \sum_{k^{0}}^{L} {}_{k\ \ k^{0}}\, r_{\,i+k\ \ k^{0}} $$

e y o e fy e coe c en r_{l} fy e fo o n y e of ne e c eq on

$$ r_{l} = \left[4 r_{\ l} \; - \sum_{k}^{L=} {}_{k}\; r_{\ l\ \ k+} \;\; r_{\ l+\ k} \right] $$

e e e coe c en $_{k}$ e en n n e nd 7 e o n e y p o c of r_{l} fo e

$$ r_{l} = \frac{\quad}{\;-\;}\,\frac{\quad}{+} \quad O\,\frac{\quad}{+\ +\ M} \qquad \text{fo} $$

$$ r_{l} = \qquad\qquad \text{fo} \qquad\qquad e $$

Example.

		Coe cients		Coe cients
	l	ı	l	ı
M = 6	-7	-2.82831017E-06	4	-2.77955293E-02
	-6	-1.68623867E-06	5	-2.61324170E-02
	-5	4.45847796E-04	6	-1.91718816E-02
	-4	-4.34633415E-03	7	-1.52272841E-02
	-3	2.28821728E-02	8	-1.24667403E-02
	-2	-8.49883759E-02	9	-1.04479500E-02
	-1	0.27799963	10	-8.92061945E-03
	0	0.84681966	11	-7.73225246E-03
	1	-0.69847577	12	-6.78614593E-03
	2	2.36400139E-02	13	-6.01838599E-03
	3	-8.97463780E-02	14	-5.38521459E-03

M plic on of ope o n e e e

VII.1 Multiplication of matrices in the standard form

e p c on of ce of C de on Zy nd nd p e do d en ope o n e nd d fo eq e o O N o N ope on n dd on po e o con o e d of e n e nd y e n o ze o e en e n e p od c e o e o d of nd e efo e

$$|| \quad - \quad || \leq \qquad (7)$$

e nd de of 7 do n ed y o e p e f e co p e [4] en e
o e one n c n d

VII.2 Multiplication of matrices in the non-standard form

e no o ne n o fo e p c on of e ope o n e non nd d fo ne o e e n e y deco p e e e n e p oce of p c on Le nd e o ope o

$$: \mathbf{L} \ \mathbf{R} \rightarrow \mathbf{L} \ \mathbf{R} \qquad (77)$$

en e non nd d fo of nd $\{A_j \ B_j \ j\}_{j \in \mathbb{Z}}$ nd $\{A_j \ B_j \ j\}_{j \in \mathbb{Z}}$ e co p e e non nd d fo $\{A_j \ B_j \ j\}_{j \in \mathbb{Z}}$ of $-$
e ec e ope o of

n y e e e 7 of o e

$$— \qquad (7)$$

e e

$$— \sum_{j}^{n} \left[A_j A_j \quad B_j \quad B_j \; j \quad A_j B_j \quad j \quad A_j \right] \qquad (7)$$

nd

$$— \; n \; n \quad \sum_{j}^{n} P_j \; B_j P_j \qquad (7)$$

e ope o n e 7 e c n on fo o n e p ce

$$A_j A_j \quad B_j \quad \mathbf{W}_j \to \mathbf{W}_j \qquad (7) \; e$$

$$B_j \; j \quad A_j B_j \quad \mathbf{V}_j \to \mathbf{W}_j \qquad (7)$$

$$j \quad A_j \quad \mathbf{W}_j \to \mathbf{V}_j \qquad (7)$$

nd e ope o n e 7

$$B_j \quad \mathbf{V}_j \to \mathbf{V}_j \qquad (7\ 7)$$

e e; — n 7 d d e f d 7 7 f

of ope on ed e c e e o o e p e e e o n e of
ope on ep p opo on o N
e e n ope o A_j B_j j n tot49 -410.315ac5 0 Td (36su8712.7097 0552

e e o n e e e

e f p c on o of ec on e econd fe o n e of e e o

VIII.1 An iterative algorithm for computing the generalized inverse

n o de o

p oced e nd e e on n e e e n ee n
fo e co p on e e pe fo ed on n p c o on nd e ed
o ne f o L N AC fo co p n e n e deco po on o e e
ed e fo o n f n

$$A_{ij} = \begin{cases} \overline{i \; j} & \\ & \end{cases}$$

e e N e cc cy e o d e o [4] e en e of X_k e o
[4] e e y e c y e o ed f e e c e on

Size $N \times N$	SVD	FWT Generalized Inverse	L_2-Error
128 $\times$ 128	20.27 sec.	25.89 sec.	3 1 $\cdot$ 10^{-4}
256 $\times$ 256	144.43 sec.	77.98 sec.	3 42 $\cdot$ 10^{-4}
512 $\times$ 512	1,155 sec. (est.)	242.84 sec.	6 0 $\cdot$ 10^{-4}
1024 $\times$ 1024	9,244 sec. (est.)	657.09 sec.	7 7 $\cdot$ 10^{-4}
...	...	...	...
2^{15} $\times$ 2^{15}	9.6 years (est.)	1 day (est.)	

Le de e e e e e o nd c n n e c f nc on
c c ope o c n e p e en ed e c en y e fo p e do d en op
e o N e c e nd e e pe fo nce of e o e epo ed
ep e y

VIII.2 An iterative algorithm for computing the projection operator on the null space.

Le con de e fo o n e on

$$X_{k+} = X_k - X_k$$

$$X = A A$$ e

e e A e d o n nd c o en

en $-X_k$ con e e o P_{null} c n e o n e e d ec y o y co n n n n n ep e en on fo $P_{null} := -A\ AA^{y}A$ e e on o co p e e ene zed n e AA^{y} e f p c on o e e e on e f fo de c of ope o e e co p e y e o fo e ene zed n e e po n d ence o e e e doe no eq e co p e y of e n e ope o on y of e po e of e ope o

VIII.3 An iterative algorithm for computing a square root of an operator.

Le de e n e on o con c o $A^{=}$ nd $A^{=}$ e e A fo p c y f d o n nd non ne e de n e ope o e con de e fo o n e on

$$Y_{l+} := Y_l - Y_l X_l Y_l$$

$$X_{l+} := -X_l \quad Y_l A$$

$$Y := -A$$

$$X := -PA$$ 7

e e c o en o e e e en e of $-A$ e n $\sqrt{\ }$ e eq ence X_l con e e o $A^{=}$ nd Y_l o $A^{=}$ By n $A := V\ DV$ e e D d on nd V n y e y o e fy o X_l nd Y_l c n e en $X_l := V\ P_l V$ nd $Y_l := V\ {}_l V$ e e P_l nd ${}_l$ e d on nd

$${}_{l+} := {}_l - {}_l P_l \quad {}_l$$

P e e

r n f f f d e y A f d f d

VIII.4 Fast algorithms for computing the exponential, sine and cosine of a matrix

e e ponen of o n ope o e ne nd co ne f nc on e on e o e con de ed n ny c c of ope o A n e c e of e ene zed n e e

X Co p n $F(u)$ n e e e e

n ec on e de e f d p e o fo co p n e e n n n e y d en e f nc on nd ep e en ed n e e An po n e p e O n y c e ene ze eo e of M Bony on e p op on of n e of o on of non ne eq on O n e c pp o c o e e no e e e pec de n e of pp c on of o

IX.1 The algorithm for evaluating u^2

e n o o co p e Le p o ec $\in$ **L R** on p ce **V**$_j$ $\in$ **Z** o

$$ j \; P_j \quad j \in \mathbf{V}_j $$

n o de o deco p e e e e e e op c e e

$$ - \; n \; \sum^{j}_{j} P_j \quad - \; P_j \quad \sum^{j}_{j} P_j \quad P_j \quad P_j \quad - P_j $$

n P_j P_j j e o n

$$ - \; n \; \sum^{j}_{j} P_j \quad j \quad j $$

o

$$ \sum^{j}_{j} P_j \quad j \quad \sum^{j}_{j} \quad j \quad j \quad n $$

n e e e no n e c on e een d en e e nd o e n e c p po e need fo o e n e n e of e o c e

Befo e p oceed n f e e con de n e p e of e n

e e e fo o n e p c e on

$$\begin{aligned} & j \; k \quad = \quad j= \quad j \; k \\ & j \; k \quad = \quad j= \quad j \; k \\ & j \; k \quad j \; k \quad = \quad j= \quad j \; k \end{aligned} \qquad 7$$

A o e pod c on e e e e zeo

p nd n e p c y n o

$$= \sum_{j}^{n} \sum_{k \in Z} d_k^j \; {}^j_k \qquad \sum_{k \in Z} {}^n_k \; {}^n_k$$

nd n 7 e o n f o e

$$= \sum_{j}^{n} {}^{j=} \sum_{k \in Z} d_k^j \; {}^j_k \; {}^j_k \qquad \sum_{j}^{n} {}^{j=} \sum_{k \in Z} d_k^j \; {}^j_k \qquad {}^{n=} \sum_{k \in Z} {}^n_k \; {}^n_k$$

On deno n

$$\begin{aligned} d_k^j \quad &= \quad j= + \; d_k^j \; {}^j_k \\ {}^j_k \quad &= \quad j= \; d_k^j \\ {}^n_k \quad &= \quad n= \; {}^n_k \end{aligned}$$

e e e

$$= \sum_{j}^{n} \sum_{k \in Z} d_k^j \; {}^j_k \qquad \sum_{j}^{n} \sum_{k \in Z} {}^j_k \; {}^j_k \qquad \sum_{k \in Z} {}^n_k \; {}^n_k$$

e no e **if the coe cient d_k^j is zero then there is no need to keep the corresponding average** ${}^j_k$ n o e o d e need o eep e e on y ne e n e e e e e e e coe c en d_k^j o p od c ${}^j_k d_k^j$ e n c n fo en cc cy

of coe c en c need o e o ed y e ed ced f e y o e n fo e p e

$$M^{j;j'}_{WWW} \quad 0 \quad - \quad j'= \int_{1}^{+1} \quad j \quad j' \quad \begin{matrix} j & j' \\ k & k' \end{matrix} \quad j \quad j' k \quad l \quad d$$

$$M^{j;j'}_{WWW} \quad 0 \quad - \quad j'= M^{j \quad j'}_{WWW} \quad - \quad 0 \quad j \quad j' \quad -$$

o e e e o n c n ed c on n e n e of coe c en con e q ence of e f c e coe c en n dec y e d nce $r - -'$ e een e e f e n e of n c n coe c en d_k^j p opo on o e n e of e o N e e n e of ope on eq ed o e e e pp n

ce nece y o o e on y o e e $_{k'}^{j}$ c co ne e n c n coe c en d_k^j o p od ce non ze o con on e efo e c en o o e on y o $_{k'}^{j}$ fo c e e e e coe c en d_k^j c $| - '| \leq$ nd e p od c $_{k'}^{j} d_k^j$ o e e e od of cc cy e n e need o o e e e on y n e ne o ood of n e

e n e of ope on fo e p nd n of e econd e n n o e e e p opo on o e n e of n c n en e nd e e e co p e e y o fo [redacted]

Remark. e o fo e on n e e e o o e e e p od c of o f nc on nce $\frac{}{4}$ − −

IX.2 The algorithm for evaluating F(u)

Le e n n n e y d en e f nc on n o de o deco p e e e e n e e op c e e

$$- \quad n \quad \sum_{j}^{n} \quad P_j \quad - \quad P_j \qquad 7$$

p nd n e f nc on n e y o e e e po n y e P d 7 7 e de p

e no e e e no e e econd de e of n e e e no e en
de e n n nd con de n e e nde of e e e n
n e o e e o n e e of M Bony e e o e o e e e
y oo e n n Bony e nce e e nde f c o j n e d
of j e y eep o e e o e e e nde y oo
no c e po n f e e n d n e n co p n
o epe ed pp c on of e o fo c en o co p e o
f nc on o e e e e e e n y c d n e n con de n n
p c y n eo e c c e z n e

efe ence

. , B A pe p e ep e en on of oo ne ope o D e Y e n e y

. , C e L een d nd o n A f d p e po e o fo
p c e on **SIAM Journal of Scienti c and Statistical Computing** e
Y e n e y ec n c epo YALeB DO o e

. e A Co en D ec e nd C .

M e e of f eq ency c nne deco po on of e nd e e ode ec n c epo e Co n n e of M e c c ence Ne Yo n e y

Y Meye Le c c en q e e onde e e e e o en q d e C MAD n e e D p ne

Y Meye nc pe d nce de e enne e e e d ope e n