Preliminary Exam
Partial Differential Equations
1:30 - 4:30 PM, Fri. Jan. 10, 2019
Room: Newton Lab (ECCR 257)

Student ID:______________________________

#	possible	score
1	25	
2	25	
3	25	
4	25	
5	25	
Total	100	

There are five problems. **Solve four of the five problems.**
Each problem is worth 25 points.
A sheet of convenient formulae is provided.

1. **Quasilinear first order equations.**
   Consider the Cauchy problem

$$\begin{aligned} u_t + (u + u^2)u_x &= 0, \qquad x \in \mathbb{R}, \quad t > 0, \\ u(x, 0) &= f(x), \qquad x \in \mathbb{R}. \end{aligned} \tag{1}$$

   (a) Suppose $f \in C^1(\mathbb{R})$ and f, f' are bounded functions. Prove that a continuously differentiable solution $u(x, t)$ to Eq. (1) exists and is unique for $x \in \mathbb{R}$, $t \in [0, t^*)$ for some $t^* > 0$.

   (b) Provide an additional, necessary condition on f for the solution to Eq. (1) to exist for all $t > 0$, i.e., for $u(x, t)$ to remain continuously differentiable for all $t > 0$.

2. **Heat Equation.**
   Let $D = (0, L) \times (0, T]$ and assume that $u \in C(\bar{D}) \cap C^2(D)$ is a solution to

$$\begin{aligned} u_t(x, t) &= g(x)u_{xx}(x, t) + F(x, t), && 0 < x < L, \quad 0 < t \le T. \\ u(x, 0) &= f(x), && 0 < x < L, \\ u(0, t) &= r(t), && 0 < t \le T, \\ u(L, t) &= s(t), && 0 < t \le T, \end{aligned} \tag{2}$$

   where $g(x) > 0$ for all $x \in (0, L)$.

   (a) Let $B = \bar{D} \setminus D$. If $F \le 0$, prove that

3. **Wave Equation.** Consider the initial boundary value problem (IBVP):

$$
\begin{aligned}
u_{tt} &= c^2 u_{xx} && x > 0,\ \ t > 0, \\
u(x, 0) &= 0 && x > 0, \\
u_t(x, 0) &= \psi(x) && x > 0, \\
u_x(0, t) &= 0 && t ;>
\end{aligned}
$$