

Department of Applied Mathematics  
 Preliminary Examination in Numerical Analysis  
 August, 2013

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**Solutions:**

**1. Root Finding.**

We want to find a function such that the iteration  $x_{n+1} = x_n - f'(x_n)/f(x_n)$  'hops about' forever within a finite interval, without ever converging. The easiest example would seem to be if the iterates form some short cycle, the simplest of all such arising if  $x_{n+1} = x_n$ , i.e.  $x_n = x_n - f'(x_n)/f(x_n)$ . Simplifying the notation by writing  $x$  in place of  $x_n$ , this will be satisfied if  $f'(x) = \frac{1}{2x} f(x)$ . We can thus choose  $f(x) = \begin{cases} c \rho^x & \text{if } x \geq 0 \\ c \rho^{-x} & \text{if } x < 0 \end{cases}$ , here  $c$  is an arbitrary constant.

**2. Numerical Quadrature.**

- (a) Let  $h$  denote the length of a single subinterval before the extrapolation is done. Including also the subinterval midpoint, the trapezoidal rule over this subinterval could have the heights at its ends and midpoint:  $T_0 = h[\frac{1}{2} \ 0 \ \frac{1}{2}]$  and, then using also the midpoint  $T_1 = h[\frac{1}{4} \ \frac{1}{2} \ \frac{1}{4}]$ .



## 5. ODEs

(a) We have

$$\mathbf{f}(t_n + h; \mathbf{y}_n + \mathbf{k}_1) = \mathbf{f}(t_n; \mathbf{y}_n) + h \frac{\partial \mathbf{f}(t_n; \mathbf{y}_n)}{\partial t} + \frac{\partial \mathbf{f}(t_n; \mathbf{y}_n)}{\partial \mathbf{y}} \mathbf{f}(t_n; \mathbf{y}_n) + O(h^2); \mathbf{y}_n + \mathbf{k}$$

**PDEs**

We look for the solution in the form

$$u(x; y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} u_{mn} \sin((m+1)x) \sin\left(n + \frac{1}{2}\right)y$$

so that

$$\frac{\partial u}{\partial y}(x; y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} u_{mn} \left(n + \frac{1}{2}\right) \sin((m+1)x) \cos\left(n + \frac{1}{2}\right)y$$

satisfies the eumann boundary at  $y = 1$ . Computing

$$u(x; y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} u_{mn} \frac{2}{(m+1)^2 + \left(n + \frac{1}{2}\right)^2} \sin((m+1)x) \sin\left(n + \frac{1}{2}\right)y ;$$

we seek an expansion of the right hand side,

$$f(x; y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f_{mn} \sin((m+1)x) \sin\left(n + \frac{1}{2}\right)y$$

so that we can set

$$u_{mn} = \frac{f_{mn}}{2 \left( (m+1)^2 + \left(n + \frac{1}{2}\right)^2 \right)} ;$$

Consider  $x_k = (k+1)\pi/M$ ,  $k = 0; \dots; M-1$  and  $y_l = (l+1)\pi/N$ ,  $l = 0; \dots; N-1$  so that

$$f(x_k; y_l) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f_{mn} \sin\left(\frac{(m+1)k\pi}{M}\right) \sin\left(\frac{(n + \frac{1}{2})l\pi}{N}\right)$$

if

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