

Department of Applied Mathematics  
**Preliminary Examination in Numerical Analysis**  
August 17, 2016 , 10 am – 1 pm.

Submit solutions to four (and no more) of the following six problems. Show all your work, and justify all your answers. No calculators allowed.

-----  
**Problems and solutions given below:**  
-----

**1. Root finding / Nonlinear equations**

Consider the scalar equation  $F(x) = 0$ . Assume  $\alpha$  is a root of the equation.

- a. Give the recursion for the Newton method for approximating a root.
- b. Give conditions on  $F(x)$  near

## Solution: Nonlinear Equations

(a) Newton:

$$x_{j+1} \equiv x_j - \frac{F(x_j)}{F'(x_j)} \quad (1)$$

(b) Let  $F(x) = x^2 - a$ ,  $F'(x) = 2x$ , and  $F''(x) = 2$ . Assume that  $x_j \neq 0$ . Then, the iteration (1) can be written as

(c) Note that  $F(a) = 0$  and solve for  $\alpha$  to get

$$\alpha - x_j = \frac{F(x_j)}{F'(x_j)} - \frac{1}{2} \frac{F''(x_j)}{F'(x_j)} (\alpha - x_j)^2.$$

Use a recurrence relation for  $\alpha_j$  in  $\sin$  and  $\cos$  to get  $\alpha_j = \alpha$ .

$$\alpha - x_{j+1} = \frac{1}{2} \frac{F''(x_j)}{F'(x_j)} (\alpha - x_j)^2.$$

$$|\alpha - x_{j+1}| \leq \frac{M}{2} |\alpha - x_j|^2$$

Since  $M$  is a constant, we can choose  $n$  such that  $M/2 < 1$ .

$$|\alpha - x_{j+1}| < |\alpha - x_j|, \quad j \geq n$$

Let  $x_0$  be chosen so that  $|\alpha - x_0| < 1/M$  and  $M/2 < 1$ .

$$|\alpha - x_1| < M |\alpha - x_0| < |\alpha - x_0|, \quad \text{and} \quad |\alpha - x_1| < 1/M.$$

By induction, this also implies  $|\alpha - x_j| < 1/M$  and  $|\alpha - x_j| < |\alpha - x_{j-1}|$  for  $j > 1$ .

$$|\alpha - x_j| < \left(\frac{M}{2}\right)^{j-1} |\alpha - x_1| < \left(\frac{M}{2}\right)^{j-1} \frac{1}{M}.$$

Thus,  $x_j \rightarrow \alpha$  and convergence is quadratic.

2.

### 3. Interpolation / Approximation

- a. Define what is meant by *cubic splines* and, for these, *natural* and *not-a-knot* conditions.
- b. Determine the *not-a-knot* cubic spline  $s(x)$  that satisfies the data  $\frac{x}{y} \left| \begin{array}{cccc} 1 & 0 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{array} \right.$ .
- c. If, at the nodes  $x = -h, 0, h$ , one has function values  $y_{-h}, y_0, y_h$  and forms a quadratic interpolant  $s(x)$ , one obtains  $s'(0) = [\frac{1}{2}y_{-h} - \frac{1}{2}y_h]/h$ , i.e. the finite difference weights can be written as  $[\frac{1}{2}, 0, -\frac{1}{2}]/h$ . It might be tempting to replace the quadratic interpolant here with a natural cubic spline (hoping to increase the approximation's order of accuracy). Work out the weights you get in this case.

#### Solution:

- a. A *cubic spline* is a cubic polynomial between adjacent nodes, and features continuous function, first and second derivative at the nodes – i.e. the third derivative may be discontinuous at the nodes. Without additional end conditions, a cubic spline will have two free parameters. A *natural* cubic spline adds the two extra conditions that  $s''(x) = 0$  at each end point. The *not-a-knot* cubic spline instead removes two 'freedoms', i.e. the cubic spline is not allowed to have a jump in its third derivative one node point in from each boundary.
- b. With four node points, and jumps in the third derivative not allowed at either of the two internal nodes, the spline becomes a single cubic, i.e. we can immediately find it, for ex., by Lagrange's or Newton's interpolation formulas. Choosing, for ex., the Newton approach, the divided difference table becomes

-1	-2			
0	-3	-1	0	
1	-4	-1	3	1
2	1	5		

from which we read off the polynomial as  $s(x) = \frac{2}{3}x^3 - \frac{1}{3}(x-1)^2 - \frac{1}{3}(x-1)x - \frac{1}{3}(x-1)x(x-1)$   
 $x^3 - 2x - 3$ .

- c. Since the spline  $s(x)$  is not discontinuous at  $x = 0$  until in the third derivative, we can write it:

$$\begin{aligned} [h, 0] & \quad a + bx + cx^2 + dx^3 \\ [0, h] & \quad a + bx + cx^2 + ex^3 \end{aligned}$$

The natural end conditions give  $2c - 6dh = 0$  and  $2c - 6eh = 0$ , resp., i.e.  $e = d$ . Enforcing the values at the nodes now give

$$\begin{aligned} a + bh + ch^2 + dh^3 &= y_{-h} \\ a &= y_0 \\ a + bh + ch^2 + dh^3 &= y_h \end{aligned}$$

Subtracting the top equation from the bottom one gives  $2bh = y_h - y_{-h}$ , and we obtain the same approximation for  $s'(0)$

#### 4. Linear Algebra

Consider the linear system  $A\underline{x} = \underline{b}$ , where  $A_{n \times m}$ ,  $\underline{x}_{m \times 1}$ ,  $\underline{b}_{n \times 1}$ .

a. Describe the three possible cases for existence and uniqueness of a solution of the linear system. Give criteria on  $A, \underline{b}$  that distinguish each case.

b. Let  $\underline{x}_{LS}$  be a minimizer of the least squares functional, that is, let

$$\|A\underline{x}_{LS} - \underline{b}\|_2 = \min_{\underline{x}} \|A\underline{x} - \underline{b}\|_2.$$

(i) Does  $\underline{x}_{LS}$  always exist? Explain your answer.

(ii) Give conditions on  $A, \underline{b}$  such that  $\underline{x}_{LS}$  is unique.

(iii) In the case of a unique solution, give an expression for the least squares solution  $\underline{x}_{LS}$ .

(iv) If there is an infinite number of solutions to the least squares problem, find the solution of minimal norm.

c. The minimal norm solution can be computed by using the singular value decomposition (SVD) of

## Solution Linear Algebra

(a) The three cases are:

(i) If  $\det(A) \neq 0$ , there exists a unique solution. (ii) If  $\det(A) = 0$  and the columns of  $A$  are linearly independent (or, equivalently,  $\text{rank}(A) = n$ ), there exists a unique solution.

(iii) If  $\det(A) = 0$  and the columns of  $A$  are linearly dependent (or, equivalently,  $\text{rank}(A) < n$ ), there exists an infinite number of solutions if:

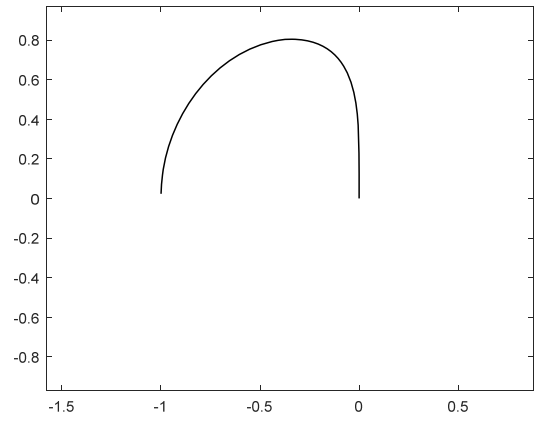
(a)  $b$  is in the column space of  $A$ . (b)  $b$  is not in the column space of  $A$ .

(c) If  $b$  is in the column space of  $A$ , the number of solutions is  $n - \text{rank}(A)$ .

(d) If  $b$  is not in the column space of  $A$ , there are no solutions.



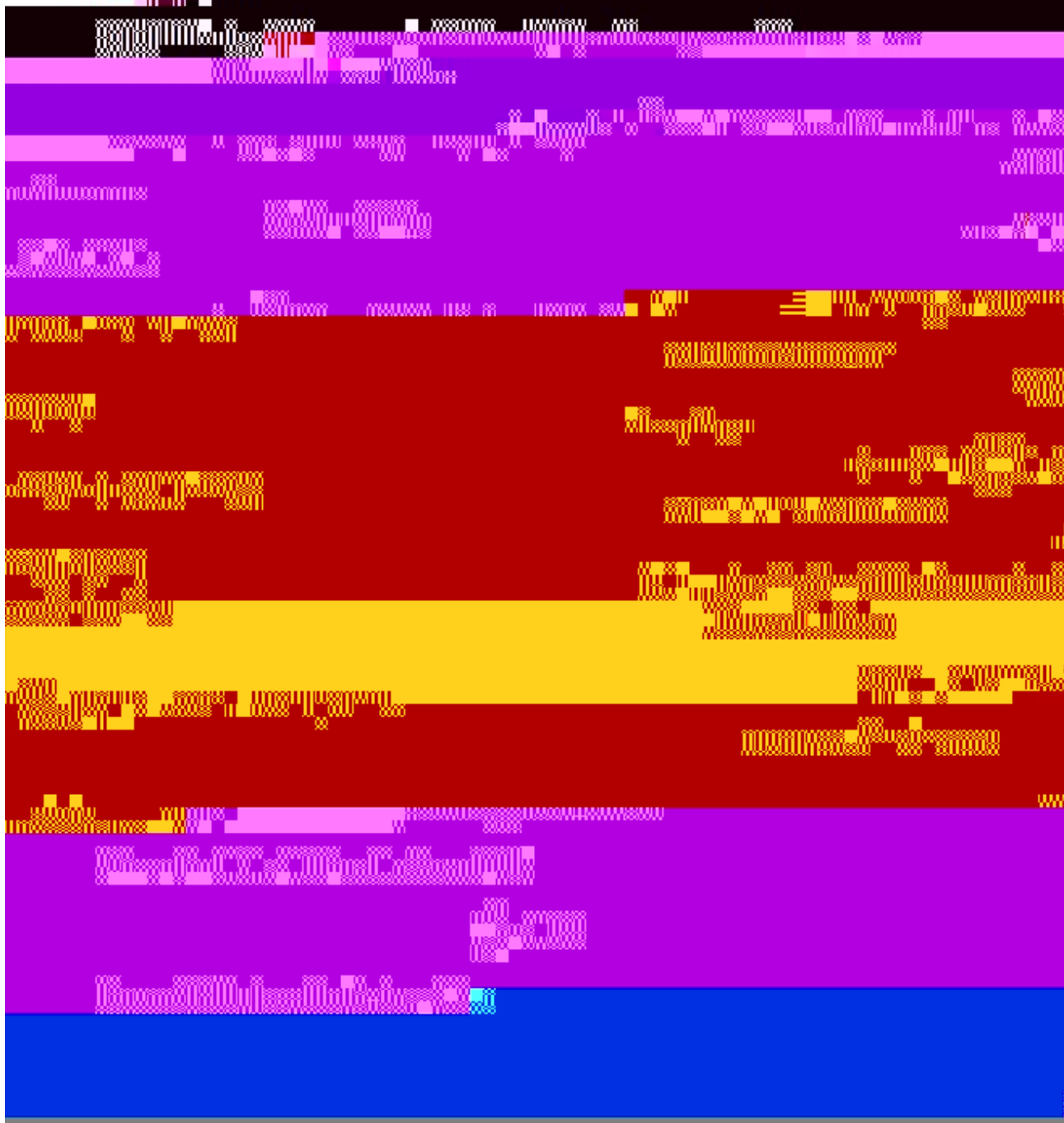








# Solution PDE



$$\frac{d}{dt} \int_{\Omega} \rho \, dx = \int_{\Omega} \rho \, dx - \int_{\Omega} \rho \, dx = 0$$

