1. Root finding

2 2

$$
\begin{aligned}
f(x, y) &= 0 \\
g(x, y) &= 0
\end{aligned}
$$

$f(x) = 0,$

f g 2 2

$x = \alpha, y = \beta$ $_n = x_n - \alpha,$ $_n = y_n - \beta$ $_{n+1}$ $_{n+1}$

$O(\ {}^{2}_{n}, \ {}^{2}_{n}).$

Solution:

$$f(x) = 0$$

2. Quadrature

(1) Consider quadrature

$$I_{quad} = \sum_{i=0}^{n} \alpha_i f(x_i), \quad x_i \in [-1,1] \tag{0.1}$$

for the integral

$$I = \int_{-1}^{1} f(x)\, w(x)\, dx,$$

where w is a positive weight in $(-1,1)$. Let

$$\Omega_{n+1}(x) = \prod_{i=1}^{n} (x - x_i)$$

denote the polynomial of degree $n+1$ associated with the (distinct) quadrature nodes $x_0, x_1, \ldots, x_n$. Prove that [illegible]

$$\int_{-1}^{1} \Omega_{n+1}(x)\, p(x)\, w(x)\, dx = 0 \tag{0.2}$$

[illegible] for any polynomial of degree less or equal to $m-1$ if and only if the [illegible]

Proof:

(1) If the quadrature formula (0.1) is exact for all polynomials of degree less or equal to $n+m$, then

$$\int_{-1}^{1} \Omega_{n+1}(x)\, p(x)\, w(x)\, dx = \sum_{i=0}^{n} \alpha_i \Omega_{n+1}(x_i)\, p(x_i)\, w(x_i) = 0$$

[illegible] equal to $n+m$. We can [illegible] Let us consider polynomial $f(x)$ of degree less or equal [illegible] write

$$f(x) = \Omega_{n+1}(x)\, \pi_{m-1}(x) + q_n(x),$$

where q_n is the remainder of the division of f by the polynomial Ω_{n+1}. [illegible] using (0.2)

$$\begin{aligned} I &= \int_{-1}^{1} f(x)\, w(x)\, dx \\ &= [\text{illegible}] \\ &= \int_{-1}^{1} q_n(x)\, w(x)\, dx \end{aligned}$$

and, by the direct evaluation,

$$\begin{aligned} I_{quad} &= \sum_{i=0}^{n} \alpha_i f(x_i) \\ &= \sum_{i=0}^{n} \alpha_i \Omega_{n+1}(x_i)\, p(x_i)\, w(x_i) + \sum_{i=0}^{n} \alpha_i q_n(x_i) \\ &= \sum_{i=0}^{n} \alpha_i q_n(x_i). \end{aligned}$$

We obtain the result by observing that the quadrature weights α_i can always be chosen to satisfy

$$I = I_{quad}$$

for an arbitrary polynomial of degree less or equal to n.

3. Interpolation / Approximation

$_n$, $n = 0, 1, 2, \ldots$ *n* *a b*
w x

$_1(\ ) \quad (\qquad) \quad (\ ) \qquad _1(\ ) \quad 0, \qquad 1, 2, 3,$

4. Linear Algebra

Let $A \in \mathbb{C}^{n\times n}$ be a symmetric complex valued matrix, $A = A^t$. It is possible to...

[redacted]

Proof:

[illegible]

[redacted]

as well as

$$A\overline{A}u = \mu A\overline{u} = \mu^2 u.$$

$$\overline{A}A\overline{u} = \mu\overline{A}u = \mu^2\overline{u}.$$

re the singular vectors and, thus, they are orthonormal. We recognize that u and $\overline{u}$ a

[illegible]

[redacted]

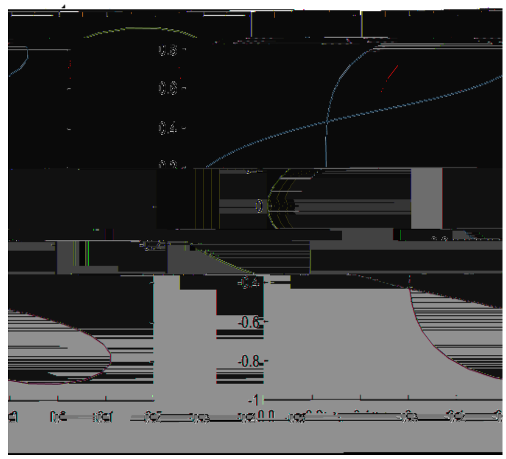
5. ODE

$$y' = f(x, y)$$

$$y_{n+1} = y_n + \frac{h}{24}\left(55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3}\right)$$

root condition

```
r  = e p(comple (0,linspace(0,2*pi)));
 i = 24*(r.^4-r.^3)./(55*r.^3-59*r.^2+37*r-9);
plot( i);
```



Proof.

(1) The even derivatives of the sine series

$$\partial_x^{2k} \partial_y^{2l} \sum_{m,n=1}^{\infty} u_{mn} \sin(\pi m x) \sin(\pi n y)$$

and, thus, is zero on the boundary. However, it is easy to see that this is

(2) One can use orthogonal polynomials, e.g., Legendre polynomials, to approximate the solution,

$$u(x,y) = \sum_{m,n=0}^{\infty} u_{mn} P_m(2x-1) P_n(2y-1).$$

This series converges rapidly since the boundary conditions for the even derivatives are not forced on the solution.

(3) We can truncate the Legendre series so that

$$u_N(x,y) = \sum_{m,n=0}^{N} u_{mn} P_m(2x-1) P_n(2y-1)$$

is an accurate approximation of the solution. We use the Legendre series for

cal scheme using such nodes in both x and y directions.