

Department of Applied Mathematics  
Preliminary Examination in Numerical Analysis  
August, 2013

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**Solutions:**

**1. Root Finding.**

(a) Let the root be  $x = r$ . We subtract  $f(x_n)$  from both sides of  $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$



#### 4. Linear algebra

- (a) Since  $A$  is an antisymmetric matrix, its eigenvalues are purely imaginary, or zero. Since it is a matrix with real entries, the roots of the characteristic polynomial come in pairs (if they are complex-valued). For odd-sized matrix these two conditions force at least one of the eigenvalues to be zero.
- (b) For even-sized matrix the product of a pair of complex-valued eigenvalues is always positive and the conclusion follows.

- (b) For an explicit multistep method, the equation for the roots of the characteristic polynomial has the form

$$(u) = u^s + \text{lower order terms} = 0:$$

Since the polynomial can be written in terms of its roots as

$$(u) = (u - u_1)(u - u_2) \cdots (u - u_s);$$

and in the region of absolute stability all roots  $|u_k| < 1$ , we conclude that, in that region, all coefficients of the polynomial are bounded (independent of  $h$ ). However, if the region of absolute stability is unbounded, then some of the coefficients will become

here

$$A = \frac{1}{2}(c)^2 - \frac{1}{2}c; \quad B = (c)^2 - 1 \quad \text{and} \quad C = \frac{1}{2}(c)^2 + \frac{1}{2}c :$$

Using  $e^{jkh_x}$   $\sum_{j=0}^{N-1}$  as an eigenvector (with index  $k = 0; \dots; N-1$ ), we compute

$$\begin{aligned} Ae^{j(j+1)kh_x} - Be^{jkh_x} + Ce^{j(j-1)kh_x} &= e^{jkh_x} [Ae^{ikh_x} - B + Ce^{-ikh_x}] \\ &= e^{jkh_x} [1 - (c)^2 + (c)^2 \cos(kh_x) - ic \sin(kh_x)] \end{aligned}$$

Computing the absolute value of the eigenvalue  $\lambda_k = 1 - (c)^2 + (c)^2 \cos(kh_x) - ic \sin(kh_x)$ , we have

$$\begin{aligned} |\lambda_k|^2 &= [1 - (c)^2 + (c)^2 \cos^2(kh_x)]^2 + (c)^2 \sin^2(kh_x) \\ &= [1 - (c)^2 \sin^2(kh_x)]^2 + (c)^2 \sin^2(kh_x) : \end{aligned}$$

Setting  $a = (c)^2$ ,  $a > 0$  and  $x = \sin^2(kh_x)$ ,  $0 \leq x \leq 1$ , as a function of  $x$  we have  $(1 - ax)^2 + ax = 1 - ax + a^2x^2$ . The condition  $a \leq 1$  implies that

$$1 - ax + a^2x^2 \leq 1 :$$

Thus, we obtain stability under the CFL condition  $c \leq 1$  or  $h_t = h_x^{-1} = c$ .