Numerical Analysis Preliminary Exam

August 15, 2011

Time: 180 Minutes

Do 4 and only 4 of the following 6 problems. Please indicate clearly which 4 you wish to have graded.

!!! No Calculators Allowed !!!

!!!Show all of your work !!!

NAME:_________________________________

For Grader Only	
1	/ 25
2	/ 25
3	/ 25
4	/ 25
5	/ 25
6	/ 25
	/100

1. Quadrature

The Chebyshev polynomials of the second kind are defined as

$$U_n(x) = \frac{1}{n+1} T'_{n+1}(x); \quad n \geq 0;$$

where $T_{n+1}(x)$ is the Chebyshev polynomial of the first kind.

(a) Using the form $T_n(x) = \cos(n\theta); \quad x = \cos(\theta); \quad x \in [-1;1];$ derive a similar expression for $U_n(x)$.

(b) Show that the Chebyshev polynomials of the second kind satisfy the recursion

$$\begin{aligned} U_0(x) &= 1 \\ U_1(x) &= 2x \\ U_{n+1}(x) &= 2xU_n(x) - U_{n-1} \end{aligned}$$

(c) Show that the Chebyshev polynomials of the second kind are orthogonal with respect to the inner product

$$< f; g >= \int_{-1}^{1} f(x)g(x)\sqrt{1-x^2}dx:$$

(d) Derive the 3 point Gauss Quadrature rule for the integral

$$I_3(f) = \sum_{j=1}^{3} w_j f(x_j) = \int_{-1}^{1} f(x)\sqrt{1-x^2}dx + \mathcal{E}_3(f);$$

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2. **Linear Algebra**

(a) Descibe the singular value decomposition (SVD) of the $m \times n$ matrix A. Include an explanation of the rank of A and how the SVD relates to the four fundamental subspaces

$\mathcal{R}(A)$	Range of A	$\mathcal{R}(A^*)$	Range of A^*
$\mathcal{N}(A)$	Nullspace of A	$\mathcal{N}(A^*)$	Nullspace of A^*

(b) Perform the SVD on the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 2 & -1 \\ 1 & 0 \end{bmatrix}$$

(c) Compute the pseudo-inverse of A (the Moore-Penrose pseudo-inverse) Leave in factored form.

(d) Find the minimal-length least-squares solution of the problem

$$A\underline{x} = \underline{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

3. **Eigenvalues** Define the $k \times k$ tridiagonal matrix

$$T_k = \begin{bmatrix} a_1 & b_2 & & & \\ c_2 & a_2 & b_3 & & \\ & c_3 & a_3 & \ddots & \\ & & \ddots & \ddots & b_k \\ & & & c_k & a_k \end{bmatrix}.$$

The characteristic polynomial of T_k is given by $p_k(\lambda) = det(\lambda I - T_k)$.

(a) Define $p_k(\lambda)$ in terms of $p_{k-1}(\lambda)$ and $p_{k-2}(\lambda)$.

(b) Show that if $c_j b_j > 0$ for $j = 2, \ldots, k$, then $p_k(\lambda) = 0$ has only real roots. (Hint: find a real similarity transformation that symmetrizes T_k.)

(c) Assume $c_j b_j > 0$ for $j = 2, \ldots, k$ and assume that the roots of $p_{k-2}(\lambda)$ separate the roots of $p_{k-1}(\lambda)$, that is, between each adjacent pair of roots of $p_{k-1}(\lambda)$, there is a root of $p_{k-2}(\lambda)$. Prove that the roots of $p_{k-1}(\lambda)$ separate the roots of $p_k(\lambda)$. (Hint: draw a picture and use the recursion.)

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5. ODE

The Forward Euler (FE) method for solving

$$y'(t) = f(t, y(t)),\ \ y(t_0) = y_0 \tag{5.1}$$

uses for each step the first two terms of its Taylor expansion, i.e.

$$y(t+h) = y(t) + h\,f(t, y(t)). \tag{5.2}$$

The *Taylor Series Method* generalizes (5.2) to include further terms in the expansion

$$y(t+h) = c_0 + c_1 h + c_2 h^2 + c_3 h^3 + \ldots + c_n h^h \ \ (+O(h^{n+1})). \tag{5.3}$$

The main interest in the Taylor series method arises when one wants extremely high orders of accuracy (typically in the range of 10-40). There are three main ways to determine (in each step) the constants $c_0,\ c_1,\ c_2, \ldots$ Many numerical text books consider only the first procedure listed below (and then dismiss the Taylor approach as generally impractical, since the number of terms more than doubles by each iteration):

Procedure 1: Differentiate (5.1) repeatedly to obtain

$$\begin{aligned} y' &= f \\ y'' &= f\tfrac{\partial f}{\partial y} + \tfrac{\partial f}{\partial t} \\ y''' &= f^2\tfrac{\partial^2 f}{\partial y^2} + f\left\{\left(\tfrac{\partial f}{\partial y}\right)^2 + 2\tfrac{\partial^2 f}{\partial t \partial y}\right\} + \left\{\tfrac{\partial^2 f}{\partial t^2} + \tfrac{\partial f}{\partial t}\tfrac{\partial f}{\partial y}\right\} \\ &\ldots . \end{aligned} \tag{5.4}$$

and then use $c_k = y^{(k)}(t)/k!$

Consider next the special case of (5.1) $y' = t^2 + y^2$. Find the first three coefficients $c_0,\ c_1,\ c_2$, starting from a general point t by means of the approaches suggested in parts (a) - (c) below. (Needless to say, you should get the same answer in all three cases)

(a) Use *Procedure 1*, as described above.

(b) Use *Procedure 2*: Note that (5.1) implies

$$\frac{d\,y(t+h)}{dh} = f(t+h, y(t+h)). \tag{5.5}$$

Substitute some leading part of (5.3) into (5.5) and equate coefficients.

(c) Use *Procedure 3*: Note that the first term of (5.3) is known. After that, each time a truncated version of (5.3) is substituted into the right hand side (RHS) of (5.5) and integrated, one gains

6. PDE

The standard second order finite difference approximation to the ODE $u''(x) = f(x)$ can schematically be written as

$$[1 \; -2 \;\; 1]\, u/h^2 = [1]\, f + O(h^2) \tag{6.1}$$

(a) Verify that the approximation

$$[1 \; -2 \;\; 1]\, u/h^2 = [1 \;\; 10 \;\; 1]\, f/12 + O(h^4) \tag{6.2}$$

indeed is fourth order accurate.

The 2-D counterparts to (6.1) and (6.2) for approximating the Poisson equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$ are

$$\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix} \frac{u}{h^2} = [1]\, f + O(h^2) \tag{6.3}$$

and

$$\begin{bmatrix} 1 & 4 & 1 \\ 4 & -20 & 4 \\ 1 & 4 & 1 \end{bmatrix} \frac{u}{6h^2} = \begin{bmatrix} & 1 & \\ 1 & 8 & 1 \\ & 1 & \end{bmatrix} \frac{f}{12} + O(h^4), \tag{6.4}$$

respectively.

(b) Sketch the structure and give the entries of the linear system that is obtained when we use (6.4) to solve a Poisson equation with Dirichlet boundary conditions on the square domain $[0,1] \times [0,1]$.

(c) In the case when $f(x,y) \equiv 0$ (i.e. solving Laplace's equation), we would expect from (6.3) and (6.4) that

$$\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix} \frac{u}{h^2} = O(h^2) \tag{6.5}$$

and

$$\begin{bmatrix} 1 & 4 & 1 \\ 4 & -20 & 4 \\ 1 & 4 & 1 \end{bmatrix} \frac{u}{6h^2} = O(h^4). \tag{6.6}$$

This is correct for (6.5) but (remarkably), the accuracy of (6.6) now jumps to $O(h^6)$. Without working through the details, outline an approach for verifying this increased order of accuracy.
