Applied Analysis Preliminary Exam

10.00am–1.00pm, August 21, 2018

Instructions. You have three hours to complete this exam. Work all five problems. Please start each problem on a new page. Please clearly indicate any work that you do not wish to be graded (e.g., write SCRATCH at the top of such a page). You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. In your proofs, you may use any major theorem on the syllabus or discussed in class, unless you are directly proving such a theorem (when in doubt, ask the proctor). Write your student number on your exam, not your name. Each problem is worth 20 points. (There are no optional problems.)

Problem 1:

(a) Assume that a function $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable. Suppose that for all $x, y \in \mathbb{R}^n$, defining the functions $g(t) = f(tx + (1-t)y)$ and $h(t) = tf(x) + (1-t)f(y)$, it holds that $(g - h)'$ is monotonically increasing for $t \in [0, 1]$. Prove that f is convex, i.e., $g(t) \le h(t)$ $\forall t \in [0, 1]$.

(b) Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be continuously differentiable, and suppose that on an open ball U containing 0, we have $\nabla \times \mathbf{F} = 0$.

(1) Let $\phi(\mathbf{x}) = \int_0^{\mathbf{x}} \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{x} \in U$. We haven't specified the path from 0 to $\mathbf{x}$. Is ϕ well-defined? Justify your answer.

(2) Show that for arbitrary points $\mathbf{x}$ and $\mathbf{y}$ in U, $\int_{\mathbf{y}}^{\mathbf{x}} (\nabla\phi) \cdot d\mathbf{r} = \int_{\mathbf{y}}^{\mathbf{x}} \mathbf{F} \cdot d\mathbf{r}$. *(This lets*

(a)