Applied Analysis Preliminary Exam

10.00am–1.00pm, August 21, 2017 (Draft v7, Aug 20)

Instructions. You have three hours to complete this exam. Work all five problems. Please start each problem on a new page. Please clearly indicate any work that you do not wish to be graded (e.g., write SCRATCH at the top of such a page). You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. In your proofs, you may use any major theorem on the syllabus or discussed in class, unless you are directly proving such a theorem (when in doubt, ask the proctor). Write your student number on your exam, not your name. Each problem is worth 20 points. (There are no optional problems.)

Problem 1:

(a) Let F be a family of equicontinuous functions from a metric space (X, d_X) to a metric space (Y, d_Y). Show that the completion of F is also equicontinuous.

(b) Let $(f_n)_{n\geq 1}$ be a sequence of functions in $C([0,1])$. Let $\|\cdot\|$ be the sup norm. Suppose that, for all n, we have

- $\|f_n\| \leq 1$,
- f_n is differentiable, and
- $\|f_n'\| \leq M$ for some $M \geq 0$.

Show that the completion of $\{f_n\}_{n\geq 1}$ is compact, and therefore that it has a convergent subsequence.

Problem 2:

Show that there is a continuous function u on $[0,1]$ such that

$$u(x) = x^2 + \frac{1}{8}\int_0^x \sin(u^2(y))\,dy.$$

Problem 3:

Let $f \in L^1(\mathbb{R})$. Show that

$$\lim_{n\to\infty}\left(\int_{\mathbb{R}} \frac{|f(x)|^n}{1+x^2}\,dx\right)^{1/n}$$

exists and equals $\|f\|_\infty$.

Problem 4:

Let $K : L^2([0,1]) \to L^2([0,1])$ be the integral operator defined by

$$Kf(x) = \int_0^x f(y)\,dy.$$

This operator can be shown to be compact by using the Arzela-Ascoli Theorem. For this problem, you may take compactness as fact.

(a) Find the adjoint operator K^* of K.

(b) Show that $\|K\|^2 = \|K^* K\|$.

(c) Show that $\|K\| = 2/\pi$. (Hint: Use part (b).)

(d) Prove that

$$K^n f(x) = \frac{1}{(n-1)!}\int_0^x f(y)(x-y)^{n-1}\,dy.$$

(e) Show that the spectral radius of K

Problem 1 Solution:

(a) This part is almost trivial. It is just here to help with part (b).

Recall that F being equicontinuous means that, for any $" > 0$, $9 \quad > 0$ such that $d_X(x; y) < \quad) \; d_Y(f(x); f(y)) < "$ holds $8\; f\; 2\; F$.

To show equicontinuity of the completion, we need only worry about the additional included functions. Let g be a function in the completion of F that was not in F to begin with. Since F is dense in the completion, we can nd an $f\; 2\; F$ that is arbitrarily close to g. In particular, chose $f\; 2\; F$ such that $d_Y(f(x); g(x)) < "=3$ $8x\; 2\; X$.

Let $" > 0$. Note that

$$d_Y(g(x); g(y)) \quad d_Y(g(x); f(x)) + d_Y(f(x); f(y)) + d_Y(f(y); g(y)):$$

Since $f\; 2\; F$, we can nd a $\quad > 0$ such that $d_Y(f(x); f(y)) < "=3$ and we are done. This gives us $d_X(x; y) < \quad) \; d_Y(g(x); g(y)) < "$.

(b) We will use the Arzela-Ascoli Throrem: Let K be a compact metric space. A subset of $C(K)$ is compact if and only if it is closed, bounded, and equicontinuous.

The completion of ff_ng, is, by de nition, closed

By the assumptions of this problem, we also have that the completion of ff_ng is bounded.

It remains to show that the completion of ff_ng is equicontinuous.

Take $" > 0$. Fix n. By the Intermediate Value Theorem, we know that, $8\; x; y\; 2\; [0; 1]$, there exists a c between x and y such that $f_n(x) \quad f_n(y) = f_n^0(c)(x \quad y)$.

Thus, we have that $f_n(x) \quad f_n(y) \quad Mjx \quad yj$.

De ne $\quad = "=M$. We then have

$$jx \quad yj < \quad) \quad jf_n(x) \quad f_n(y)j < ":$$

Note that this is independent of the choice of n.

Thus, the family of functions ff_ng is equicontinuous.

By part (a) we know then that the completion of this family is equicontinuous.

By the Arzela-Ascoli Throrem, we then have that the completion of ff_ng is compact, as desired.

Problem 2 Solution:

We will use the Contraction Mapping Theorem: If $T : X \; ! \; X$ is a contraction mapping on a complete metric space $(X; d)$, then T has exaclty one xed point. (i.e. There is exactly one $x\; 2\; X$ such that $T(x) = x$.)

De ne

$$Tu(x) = x^2 + \frac{1}{8} \int_0^x \sin(u^2(y))\, dy:$$

Note that T maps $C([0; 1])$ functions to $C([0; 1])$ functions. Since $C([0; 1])$ is complete with respect to the sup norm $jj \quad jj_1$, Zthe contraction mapping theorem aorem aorem aorem u.504how tha n g), sin (8 x8

By the mean value theorem, we know that there is some $s \in [0,1]$ such that

$$\frac{\sin u - \sin v}{u - v} = \cos s \le 1$$

so

$$|\sin u(y) - \sin v(y)| \le |u(y) - v(y)|.$$

So, we have that

$$\begin{aligned}
\|Tu - Tv\|_\infty &\le \tfrac{1}{8} \sup_{0 \le x \le 1} \int_0^x |u^2(y) - v^2(y)|\, dy \\
&= \tfrac{1}{8} \sup_{0 \le x \le 1} \int_0^x |u(y) + v(y)|\, |u(y) - v(y)|\, dy \\
&\le \tfrac{1}{8} \sup_{0 \le x \le 1} \int_0^x (|u(y)| + |v(y)|)\, |u(y) - v(y)|\, dy
\end{aligned}$$

Since u and v are assumed to be continuous functions on the closed bounded interval $[0,1]$, they are bounded on $[0,1]$. Suppose that they are bounded by $M > 0$. Then

$$\begin{aligned}
\|Tu - Tv\|_\infty &\le \tfrac{2M}{8} \sup_{0 \le x \le 1} \int_0^x |u(y) - v(y)|\, dy \\
&\le \tfrac{M}{4} \textstyle\int_0^1 |u(y) - v(y)|\, dy \le \tfrac{M}{4} \|u - v\|_\infty \int_0^1 dy \\
&= \tfrac{M}{4} \|u - v\|_\infty
\end{aligned}$$

This may or may not be a contraction, depending on the value of M, but, we are trying to show **existence** of a solution in $C([0,1])$. If we can show existence of a solution on some subset of $C([0,1])$, we are done. So, let's limit our search to the set of continuous functions on $[0,1]$ that are bounded, in the uniform norm, by some **fixed** constant M such that $M < 4$. **Fix** such an M and define the space

$$\mathcal{C} := \{u \in C([0,1]) : \|u\|_\infty \le M\} \subset C([0,1]).$$

Note that this is a closed (and non-empty!) subset of the complete $C([0,1])$ and is therefore complete. Furthermore, M can be chosen so that $T : \mathcal{C} \to \mathcal{C}$.
Thus, we have a contraction maping on a complete space (that is a subspace of the space of interest). By the Contraction Mapping Theorem, there exists a unique fixed point $u \in \mathcal{C} \subset C([0,1])$, which is a solution to the problem.

Problem 3 Solution:
This is trivial if $\|f\|_\infty = 0$. So, let us consider the case where $\|f\|_\infty > 0$.
Note that

$$\left(\int_{\mathbb{R}} \frac{|f(x)|^n}{1 + x^2}\, dx\right)^{1/n} \le \|f\|_\infty \left(\int_{\mathbb{R}} \frac{1}{1 + x^2}\, dx\right)^{1/n} = \|f\|_\infty\, \pi^{1/n} \to \|f\|_\infty \qquad \text{(S1)}$$

as $n \to \infty$.
On the other hand, by definition of $\|f\|_\infty$, for any $0 < \varepsilon < \|f\|_\infty$, there exists an $A \subset \mathbb{R}$ (with positive Lebesgue measure) such that $|f(x)| > \|f\|_\infty - \varepsilon \ \forall x \in A$.
Thus, we have

$$\int_{\mathbb{R}} \frac{|f(x)|^n}{1 + x^2}\, dx \ge \int_A \frac{|f(x)|^n}{1 + x^2}\, dx \ge (\|f\|_\infty - \varepsilon)^n \int_A \frac{1}{1 + x^2}\, dx.$$

Note that $\int_A \frac{1}{1+x^2}\, dx$ is strictly positive. Call it $c > 0$.

For all n, we now have

$$\int_{\mathbb{R}} \frac{|f(x)|^n}{1+x^2}$$

(c)

so we have that

$$[(n-1)!]^{1/n} \sim (\sqrt{2\pi})^{1/n}(n-1)^{1-1/(2n)}e^{1/n-1}$$

which goes to ∞ as $n \to \infty$.

In conclusion, the spectral radius is

$$r(K) = \lim_{n \to \infty} ||K^n||^{1/n} = 0;$$

as desired.

Problem 5 Solution: