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The analysis of games and sports as complex systems can give insights into the dynamics of human competition and has been proven useful in soccer, basketball, and other professional sports. In this paper, we present a model for dodgeball, a popular sport in U.S. schools, and analyze it using an ordinary differential equation (ODE) compartmental model and stochastic agent-based game simulations. The ODE model reveals a rich landscape with different game dynamics occurring depending on the strategies used by the teams, which can in some cases be mapped to scenarios in competitive species models. Stochastic agent-based game simulations confirm and complement the predictions of the deterministic ODE models. In some scenarios, game victory can be interpreted as a noise-driven escape from the basin of attraction of a stable fixed point, resulting in extremely long games when the number of players is large. Using the ODE and agent-based models, we construct a strategy to increase the probability of winning.

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I. I D C I

Games and sports are emerging as a rich test bed to study the dynamics of competition in a controlled environment. Examples include the analysis of passing networks [1,2] and entropy [3] in soccer games (see also Ref. [4] for a discussion on data-driven tactical approaches), scoring dynamics [5–7], and play-by-play modeling [8,9] in professional sports such as hockey, basketball, football, and table tennis, penalty kicks in soccer games [10], and serves in tennis matches [11]. Here we explore the dynamics of *dodgeball*, where the number of players playing different roles changes dynamically and ultimately determines the outcome of the game. While modeling dodgeball might seem like a very specific task, it is a relatively clean and well-defined system where the ability of mean-field techniques [12,13] to describe human competition can be put to the test. In addition, it complements ongoing efforts to quantify and model dynamics in sports and games [1–11].

In this paper, we present and analyze a mathematical model of dodgeball based on both agent-based stochastic game simulations and an ordinary differential equation (ODE)–based compartmental model. By analyzing the stability of fixed points of the ODE system, we find that different game dynamics can occur depending on the teams' strategies: one of the teams achieves a quick victory, either team can achieve a victory depending on initial conditions, or the game evolves into a stalemate. For the simplest strategy choice, these regimes can be interpreted in the context of a competitive Lotka-Volterra model. Numerical simulations of games based on stochastic behavior of individual players reveal that the stalemate regime corresponds to extremely long games with large fluctuations. These long games can be interpreted as a noise-driven escape from the basin of attraction of the stable stalemate fixed point l are commonly observed in dodgeball games (see Fig. 2). Using both the stochastic ODE models, we develop greedy strategy demonstrate it using stochastic simulations.

The structure for the paper is as follows. In Sec. II, we describe the rules of the game we will analyze. In Sec. III, we present analyze compartment-based model of dodgeball. In Sec. IV, we present stochastic numerical simulations of dodgeball games a7d compare these with the predictions of the compartmental model. We then discuss the notion of strategy in the context of this stochastic model. Finally, we present our conclusions in Sec. V.

II. DE C I F D DGEBALL

In this paper, we consider the following variant played often in elementary schools in the United States (sometimes called *prison dodgeball*). Two teams (team 1 and team 2) of N players each initially occupy two zones adjacent to each other, which we will refer to as court 1 and court 2 (see Fig. 1). Players in a court can throw balls at players of the opposite team in the other court. If a player in a court is hit by such a ball, they move to their respective team's *jail*, an area behind the opposite team's court. A player in a court may also throw a ball to a player of their own team in their jail, and if the ball is caught, the catching player returns to their team's court (illustrated schematically in Fig. 3). We denote the number of players on team i that are in court i and jail i by X_i and Y_i, respectively. Team i loses when $X_i = 0$. For simplicity, we assume there are always available balls and neglect the possibility that a player catches a ball thrown at them by an enemy player.

In practice, games often last a long time without any of the teams managing to send all the enemy players to jail. Because of this, such games are stopped at a predetermined

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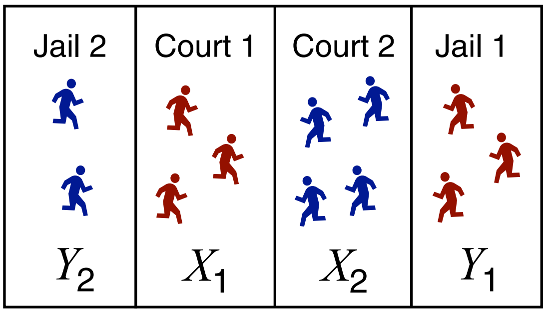

FIG. 1. (a) Setup of dodgeball court. Players in team *i* make transitions between court *i* and jail *i*, and team *i* loses when there are no players in court *i*.

time and the winner is decided based on other factors (e.g., which team has more players on their court). An example of this is in Fig. 2, which shows the numbers of players in courts 1 and 2, X_1 and X_2, during two fifth-grade dodgeball games in Eisenhower Elementary in Boulder, Colorado. The values of X_1 and X_2 seem to fluctuate without any team obtaining decisive advantage. The games continued after the time interval shown and were eventually stopped. Our subsequent model and analysis suggests that this stalemate behavior is

TABLE I. Notation used in the dodgeball model, Eqs. (5) and (6).

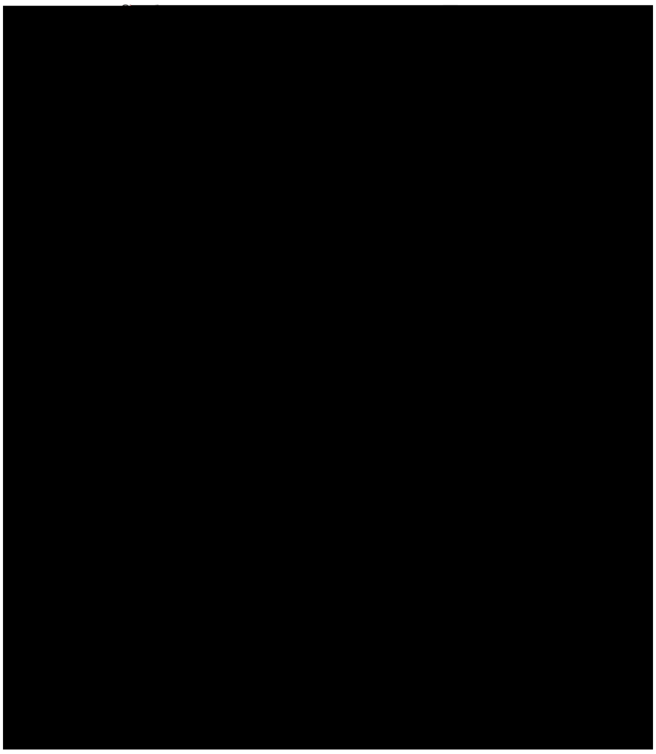

FIG. 7. Simulations of games with the same constants as Fig. 4. Trajectories (X_1, X_2) have stochastic fluctuations on top of the deterministic flow of Fig. 4. The "stalemate" regime (top left) results in long, back-and-forth games.

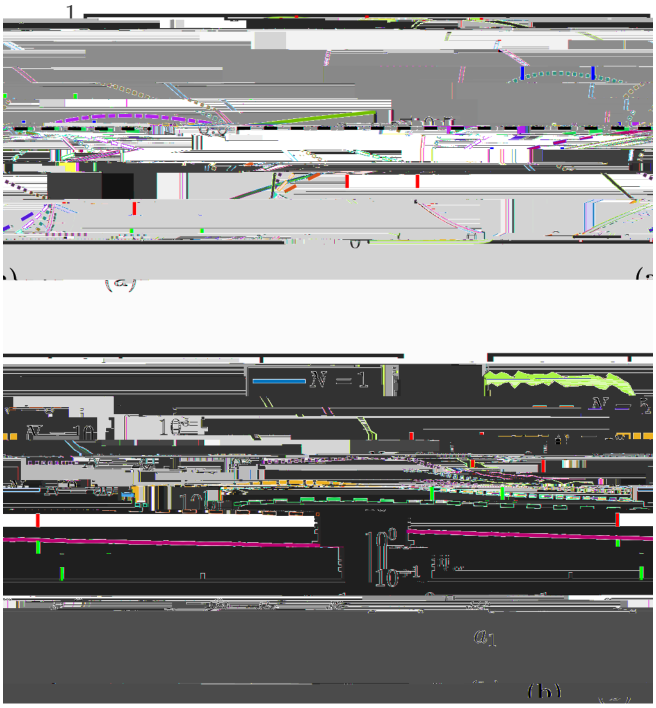

FIG. 9. (a) Probability that team 1 wins a game P_1 as a function of a_1 with $c = 2/3$ and $a_2 = 3/4$ for $N = 1, 5, 10, 20$, and 50 (solid blue, dashed orange, dashed dotted yellow, dotted purple, and solid light green lines, respectively). The dashed red lines mark bifurcations in the deterministic dynamics (see text), and the dashed horizontal line indicates $P_1 = 1/2$. The leftmost region corresponds to the "stalemate" regime, which leads to long games. The middle region represents "team 1 wins," which can be noted by the large values of P_1 for large values of N. The right region is the "competitive" region in the deterministic model noted by mixed values of P_1 and quicker games. (b) Average duration of games (in dimensionless time $\tau = \lambda N k_j t$) with the same parameters as in the bottom panel. The duration of games in the "stalemate" regime increases with N. The shaded area around the green curve represents three standard deviations.

for $a_2 < a_1$. We note that for very small N (e.g., $N = 1, 5$), the predictions of the deterministic theory break down. This can be understood in the limiting case $N = 1$ (solid blue curve), gion labeled "competitive" in Fig. 5(a) is strongly affected by stochastic fluctuations in the agent-based model. A treatment of the effect of finite-size fluctuations using the so-called "linear noise approximation" [16] could allow one to study quantitatively the escape from the basin of attraction of the stalemate fixed point, but we do not attempt this approach here.

B. H a

In the example treated in the previous sections, the probability that a player in team i decides to throw a ball to an enemy player instead of rescuing a teammate from jail, $F_i(X_1, X_2)$ is fixed throughout the game at the value a_i. In reality, players may adjust this probability in order to optimize the probability of winning. In this section, we will develop a heuristic greedy strategy with the goal of trying to optimize victory. For this purpose, it is useful to define the quantities H_i as

$$H_1 = \frac{X_1}{X_1 + X_2}, \quad H_2 = \frac{X_2}{X_1 + X_2}. \tag{12}$$

These quantities have the advantage that they are normalized between 0 and 1, with $H_i = 0$ ($H_i = 1$) corresponding to a loss (victory) by team i. In addition, H_i corresponds to the probability that team i will throw a ball next, and therefore it is a good indicator of how much control team i has. Therefore, it is reasonable for team i to apply a strategy to increase H_i. To develop such a strategy, we define H_i and H_i^+ as the values of H_i before and after a ball is thrown. Similarly, we define X_i and X_i^+ as the values of X_i before and after a ball is thrown. For definiteness, we will present the strategy for team 1, and the strategy for team 2 will be similar. The basis of the strategy is to choose the value of $F_1(X_1, X_2)$ that maximizes the expected value of H_1^+, $\mathbb{E}[H_1^+]$. Since F_1 is the probability that the ball is thrown at enemy players, p_e is the probability that such a ball actually hits an enemy player, $1 - F_1$ is the probability that the ball is thrown at a teammate in jail, and p_j is the probability that such a ball is successful in rescuing a teammate, the expected value of H_1^+ is given by

$$\mathbb{E}[H_1^+] = F_1\left[\frac{X_1}{X_1 + X_2 - 1}p_e + \frac{X_1}{X_1 + X_2}(1 - p_e)\right] + (1 - F_1)\left[\frac{X_1 + 1}{X_1 + X_2 + 1}p_j + \frac{X_1}{X_1 + X_2}(1 - p_j)\right], \tag{13}$$

which can be rewritten as

$$\mathbb{E}[H_1^+] = A + \frac{B}{X_1 + X_2}F_1, \tag{14}$$

where

$$B = \left[\frac{X_1^t}{X_1^t + X_2^t - 1}p_e - \frac{X_2^t}{X_1^t + X_2^t + 1}p_j\right] \tag{15}$$

and A is independent of F_1.

Since Eq. (14) is linear in F_1, it is maximized by choosing $F_1 = 1$ when $B > 0$ and $F_1 = 0$ when $B < 0$. Therefore, the choice of F_1 that maximizes the expected value of H_1^+, $F_1^{\square}$, is

$$F_1^{\square} = \begin{cases} 1, & \frac{X_1}{X_1+X_2-1}p_e(X_2) \geqslant \frac{X_2}{X_1+X_2+1}p_j(N - X_1), \\ 0, & \text{otherwise.} \end{cases} \tag{16}$$

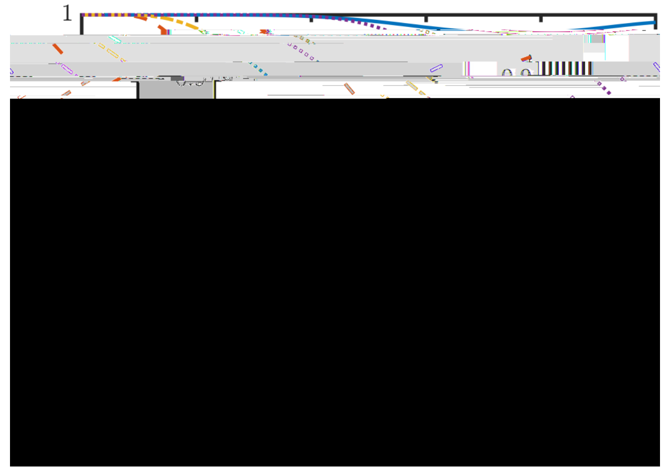

FIG. 11. Probability of team 1 winning with the heuristic strategy F_1 against a fixed strategy a_2. Number of players in each game is set to $N = 20$.

When X_1, X_2 □ 1, the strategy simplifies to

$$F_1^{\square} \approx \begin{cases} 1, & X_1 p_e(X_2) \geqslant X_2 p_j(N - X_1), \\ 0, & \text{otherwise.} \end{cases} \tag{17}$$

We note that this can also be derived by maximizing dH_1/dt by using Eqs. (3) and (4). Furthermore, for the case considered in Secs. III and IV, where $p_e(X_i) = k_e X_i$ and $p_j(Y_i) = k_j Y_i$, the strategy reduces to

$$F_1^{\square} = \begin{cases} 1, & k_e X_1 \geqslant k_j(N - X_1), \\ 0, & \text{otherwise.} \end{cases} \tag{18}$$

For example, when $k_e = k_j$ (i.e., the probability of success in hitting an enemy player is the same as the probability of succeeding in rescuing a teammate from jail), the strategy for team 1 consists in trying always to rescue teammates from jail 1 when the majority of team 1 player's are in jail 1, and in trying to hit players from team 2 when the majority of team vealed a rich dynamical landscape. Depending on teams' strategies, the dynamics and outcome of the game are determined by a combination of the stability of the fixed points

In practice, we stop this iteration when $j > J_{\max} = 256$. The iteration described by Eq. (A7) uses repeated nonsparse matrix multiplications, while Eq. (A4) uses faster sparse matrix-vector products. However, since games can be extremely long in the stalemate regime, the method described by Eq. (A7) is still faster in that regime. We choose the values $J_{\max}$ and $K_{\max}$ such that in practice Eqs. (A4) and (A7) take similar amounts of time in the stalemate regime.

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