

# Synchronization in large directed networks of coupled phase oscillators

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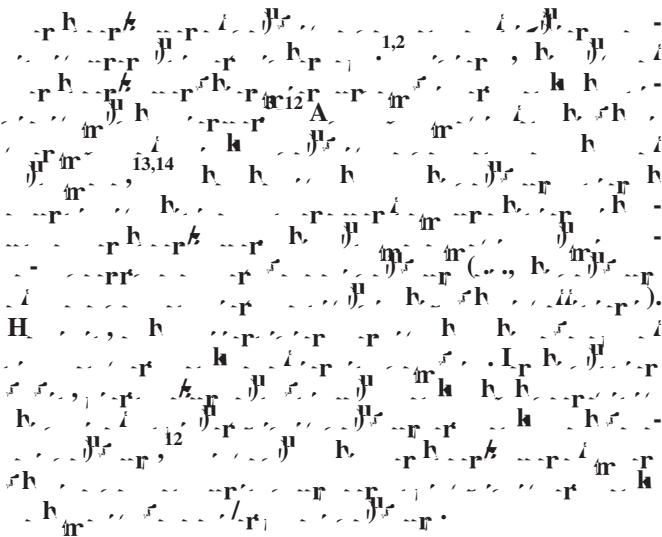
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We study the emergence of collective synchronization in large directed networks of heterogeneous oscillators by generalizing the classical Kuramoto model of globally coupled phase oscillators to more realistic networks. We extend recent theoretical approximations describing the synchronization in large, undirected networks of coupled phase oscillators to the case of directed networks. We also consider the case of networks with mixed positive-negative coupling strength. We compare our theoretical numerical simulations and find good agreement. — 2005 American Institute of Physics. [DOI: 10.1063/1.2148388]



## I. INTRODUCTION

The classical Kuramoto model<sup>13,14</sup> describes a collection of globally coupled phase oscillators that exhibit a transition from incoherence to synchronization as the coupling strength is increased past a critical value. Since real world networks typically have a more complex structure than all-to-all coupling,<sup>15,16</sup> this natural network has effective interaction structure has an impact on the synchronization transition. In Ref. 12, we studied the Kuramoto model allowing general connectivity of the nodes, and found that for a large class of networks there is still a transition to global synchronization as the coupling strength exceeds a critical value  $k_c$ . We found that the critical coupling strength depends on the largest eigenvalue of the

adjacency matrix  $A$  describing the network connectivity. We also developed a general approximation describing the behavior of an order parameter measuring the coherence of the transition. This paper addresses the case in which  $A_{nm} = A_{mn}$ . 0, has a directed network in which the coupling ends to reduce the phase difference of the oscillators.

Most networks considered in applications are directed,<sup>15,16</sup> which implies an asymmetric adjacency matrix,  $A_{nm} \neq A_{mn}$ . Also, in some cases the coupling between oscillators might differ in magnitude or phase, which can be represented by allowing the coupling term between the oscillators to be negative,  $A_{nm} < 0$ . The effect of the presence of directed and mixed positive-negative connections can have on synchronization, therefore, of interest. Here we show how our previous theory can be generalized to account for the effect of the adjacency matrix or the effect of the negative connections are particularly interesting and compare our theoretical approximation with numerical observations.

This paper is organized as follows. In Sec. II we review the results of Ref. 12 for undirected networks with positive

man the ergodicity of coupled phase oscillators. This equation can be modeled by the equation

$$\dot{\theta}_n = \omega_n + k \sum_{m=1}^N \sin(\theta_m - \theta_n)$$

A eraging o er he freq encie , one ob ain he *frequency distribution approximation* (FDA)

$$r_n = k \sum_m A_{nm} r_m \int_1^1 g(zkr_m) \sqrt{1 - z^2} dz. \quad (13)$$

The al e of he cri cal co pling reng h can be obtained from he freq enc di rib ion appro imation b le ing  $r_n \rightarrow 0^+$ , prod cing

$$r_n^{(0)} = \frac{k}{k_0} \sum_m A_{nm} r_m^{(0)}, \quad (14)$$

here  $k_0 \equiv 2/[g(0)]$ . The cri cal co pling reng h h corre pond o

$$k_c = \frac{k_0}{2}, \quad (15)$$

here i he large eigen al e of he adjacenc ma ri  $A$  and  $r^{(0)}$  i propor onal o he corre ponding eigen ec or of  $A$ . B con idering per rba ion from he cri cal al e a  $r_n = r_n^{(0)} + r_n$ , e panding  $g(zkr_m)$  in Eq. (13) o econd order for mall arg men , m l ipl ing Eq. (13) b  $r_n^{(0)}$  and , m ming o er  $n$ , e obtained an e pre ion for he order param e er pa he ran ion alid for ne ork i h rela i el homogeneo degree di rib ion <sup>17</sup>

$$r^2 = \left( \frac{1}{k_0^2} \right) \left( \frac{k}{k_c} - 1 \right) \left( \frac{k}{k_c} \right)^3, \quad (16)$$

for  $0 < (k/k_c) - 1 \ll 1$ , he2r4115.078hj/F6.768598406.2813Tm2764444111..9.9789j/F599Tc-307.9h37859.di rib ion 6.913Tm276444



$$r = \sum_{n=1}^N r$$

hand, the TAT and the relative from direct numerical solution of Eq. (1) show dependence on the realization. Since the FDA and MFT incorporate the realization of the connection  $A_{nm}$ , both the frequencies, we interpret the observed realization dependence of the TAT and the direct solution of Eq. (1) as indicating that the latter dependence is due primarily to variation in the realization of the frequencies rather than to variation in the realization of  $A_{nm}$ .

We have for our example  $N=1500$  and  $s=2/15$  implying that the average degree  $d^{in} \approx d^{out} \approx 200$ . This for comparison purposes, we generated an undirected network as follows: Starting with a  $E_{a9F54825ek-2}$  in the TDform realization a directed

the adjacent matrix is independent chosen to be 1 with probability  $s$  and 0 with probability  $1-s$ , and the diagonal elements are zero.) Even though the network constructed in this way is directed, for most nodes  $d_n^{in} \approx d_n^{out}$ . For  $N=1500$  and  $s=2/15$ , Fig. 1(a) shows the average of the order parameter  $r^2$  obtained from numerical solution of Eq. (1) averaged over realizations of the network and frequencies (triangle), the frequency distribution approximation (FDA, solid line), and the mean field theory (MFT, long dashed line) as a function of  $k/k_c$ , where the relative for the FDA and the MFT are averaged over the network realization (note, however, that the FDA and the MFT do not depend on the frequency realization). (The perturbation theory Eq. (16) agreed with the frequency distribution approximation and is left out for clarity.) The error bars correspond to one standard deviation of the sample of realizations. We note that the larger error bars occur after the transition. When the values of the order parameter are averaged over realizations of the network and the frequencies, the relative shows good agreement with the frequency distribution approximation and the directed mean field theory.

In order to describe the order parameter dependence in Fig. 1(b) the order parameter  $r^2$  obtained from numerical solution of Eq. (1) for a particular realization of the network and frequencies (box), the time averaged theory (short dashed line), and the frequency distribution approximation (solid line) as a function of  $k/k_c$ . As can be observed from the figure, in contrast with the time averaged theory, the frequency distribution approximation deviates from the numerical solution (box) by a small but noticeable amount. This behavior is observed for the other realization as well. We note that the FDA and MFT relative are in fact identical for all network realizations. On the other

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he order paramer ob ained from numerical ol ion of Eq.  
(1). The direct ed per , rba ion heor gi e a good appro i-  
ma ion for mall al e of  $k$  clo e o  $k_c$ , a e pec ed. On he  
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ion poin hich i maller han he one ac , all ob er ed.

When numerical solution of Eq. (32) by iteration of Eq. (33), on some occasions a period of orbital bifurcation is found in each of the desired points. If we denote the left hand side of Eq. (33) by  $z_n^{j+1}$  and the right hand side by  $f(z_n^j)$ , we found convergence to a fixed point is facilitated by replacing the right hand side by  $[z_n^j + f(z_n^j)]/2$  and finding the fixed point of this modified system.

In this example, a low coupling strength [ $k/k_c \approx 4$ , here  $k_c$  is computed from Eq. (37)] the order parameter computed from numerical solution of Eq. (1) is smaller than has obtained from the TAT and FDA. As  $k$  increases, however, the TAT and FDA theories capture the asymptotic value of the order parameter  $r$ . We note that in this case the asymptotic value is larger than has corresponding to phase locking [i.e., the one obtained by setting  $\dot{\theta}_n = 0$  in Eq. (35),  $r \approx 0.54 - 0.46 = 0.08$ ], which is indicated by a horizontal dashed line in Fig. 4, and much smaller than  $r = 1$ , the value corresponding to no frustration [i.e.,  $\dot{\theta}_n = 0$  for  $A_{nm} \rightarrow 0$  and for  $A_{mm} = 0$  in Eq. (35)]. The small scale of the horizontal axis indicates the fact that we are plotting  $r^2$ , and our definition of the order parameter which assigns a value of 1 to a nonfrustrated configuration. The small value of the order parameter indicates a strong frustration.

We note that in this example, in contrast to the example discussed so far, there is a variation in the value of the order parameter predicted by the FDA for different realizations of the network. This indicates that, as the expected value of the coupling strength  $A_{nm}$  becomes small (i.e.,  $|q - 1/2|$  small), frustration of the realization of the network becomes noticeable. Although the value predicted by the FDA and TAT depend on the realization of the network and frequency, we note for  $k/k_c \approx 6$  that the value of the order parameter observed for the numerical simulation of the corresponding realization. An illustration of this is plotted in Fig. 5 the value of  $r^2$  obtained from the TAT (bar) and the value of  $r^2$  obtained from the FDA (diamond) versus the value obtained from numerical solution of Eq. (1) for  $k/k_c = 8$ . Each point corresponds to a given realization of the network, with the error bars representing the error in the frequency. The ellipse surrounding the bar (TAT data) has a vertical and horizontal half-width corresponding to the an-

standard deviation of  $r^2$  (TAT) and  $r^2$  (simulation) for the ensemble of realizations. The half-width of the horizontal bar on the diamond (FDA data) indicates the standard deviation of  $r^2$  (simulation).



lation in network with a much larger number of connections per node, the effect of correlation would likely be reduced.

We also considered a case in which the adjacency matrix is asymmetric and has mixed positive-negative connections. For  $N=1500$  nodes, we considered an adjacency matrix being a nondiagonal entries of 1, -1, and 0 with probabilities  $8/45$ ,  $4/45$ , and  $11/15$ , respectively. The latter probability yielded an expected number of connections of 400. Other theoretical work is difficult in this case, and, since the results are similar to those in Fig. 3, we do not show them. In this case there is no guarantee that there is a real eigenvalue [a needed for estimating the critical coupling strength in Eq. (15)], or that the large real eigenvalue (if there is one) has the large real part. Numerically, we indeed have found a real eigenvalue and that, furthermore, it is well separated from the large real part of the remaining eigenvalues (see Fig. 6). We also find that for other values of  $q$  provided  $|q - \frac{1}{2}|$  is not too small. We provide a discussion of this issue and how the

of the non-ero en rie being cho en randoml (e.g., in the mme ric ca e, the po i ion of the non-ero en rie i cho en hen con r cing the ne ork , ing the con g ra ion model), and heir al e being al o de rmined randoml from a gi en probabili di rib, ion (e.g., 1 i h probabili  $q$  and 1 i h probabili 1  $q$ ). Or r in ere i foc, ed on the gap be een the large real eigen al e (if here i one) and the large real par of the o her eigen al e . In Ref. 23 the pec r m of cer ain large par e ma rice i h a erage eigen al e 0 and ro , m  $\sum_{m=1} A_{nm}=1$  a de cribed and a he ric anal ical approach a propo ed. U ing re , l for ma rice i h ero mean Ga ian random en rie ,<sup>24</sup> Ref. 23 predic ha the pec r m of the non-Ga ian random ma rice he con ider con i of a ri ial eigen al e =1 i h the remaining eigen al e di rib, ed , niforml in a circle cen ered a the origin of the comple plane i h radi

$$= \sqrt{N} , \tag{A1}$$

here <sup>2</sup> i the ariance of the en rie of the ma ri . We nd ha hi approach al o , cceed in de cribing the pec r m of the ma rice in o r e ample . In o r ca e, the diagonal en rie are 0, o ha the a erage eigen al e i al o 0 a in Ref. 23. We nd ha here i al a large real eigen al e appro ima el gi en b the mean eld al e

$$= \langle \tilde{d}^2 \rangle / \langle \tilde{d} \rangle \tag{A2}$$

( ee Ref . 12 and 25), here  $\tilde{d}_n = \sum_{m=1}^N A_{nm}$  and  $\langle \tilde{d}^2 \rangle = \sum_{n=1}^N \tilde{d}_n^2$ , hich in the ca e con idered in Ref. 23 red ce o =1. We al o n, mericall con rm ha the remaining eigen al e are , niforml di rib, ed in a circle of radi a de cribed in Ref. 23. Thi i ill, ra ed in Fig. 6.

Th, for  $N \gg 1$  if here i a gap of i e be een the large real eigen al e and real par of the re of the eigen al e pec r m. U ing Eq . (A1) and (A2) i can be ho n ha , for ne ork i h large eno, gh n, mber of con nec ion per node or i h eno, gh po i i e (or nega i e) bia in the co pling reng h, here i a ide epara ion be een the large eigen al e and the large real par of the remain-

ing eigen ec or . For mme ric ma rice , imilar re , l appl (i.e., the b lk of the pec r m of the ma ri  $A$  can be appro ima el ob ained a de cribed abo e , ing Wigner' emicircle la ).

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