Approximations and Fast Algorithms

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ABSTRACT

The key element in the design of fast algorithms in n

. BASES   ITH SMOOTH FUNCTIONS.

The original goal of constructing wavelets was to find a smooth generalization of the Haar basis. It is, therefore, not surprising that some definitions of wavelets include the requirement that the basis functions are at least continuous. The smoothness is desireable to ensure sufficiently fast decay of the

3. POLYNOMIAL

Thus, up to rescaling, we have an interpolating basis of scaling functions.

The Legendre nodes $x_0, \ldots, x_{k-1}$ are not uniform and concentrate near the boundary. The condition numbers of boundary operators do not become large and the boundary conditions can be succesfully used for polynomials up to degree of about 30. For higher degrees, the concentration of the nodes near the ends of the interval start to create problems since the distances between the nodes are of order $O(1/\ ^2)$. Such spatial concentration of nodes also causes difficulties in time evolution schemes by restricting the size of time steps. Yet, the range of degrees is wide enough for practical purposes and significantly extends what can be done with mutiresolution bases PDE solvers.

Let us define $\mathbf{V}_n^k$ as a space of piecewise polynomial functions,

$$
\mathbf{V}_n^k = \{f: \quad \text{the restriction of } f \text{ to the interval } (2^{-n}l,\ 2^{-n}(l+1)) \text{ is} \\
\text{a polynomial of degree less than} \quad, \text{ for } l = 0, \ldots, 2^n - 1, \\
\text{and } f \text{ vanishes elsewhere}\}. \tag{6}
$$

Let $\phi_0, \ldots, \phi_{k-1}$ be a basis of $\mathbf{V}_0^k$, then the

The scale consistent derivative operator on $\mathbf{V}_0^k$ is constructed as a transition matrix R^n between the coefficients of the expansion of the function and that of its derivative. The scale consitency means that on $\mathbf{V}_n^k$ the transition matrix is simply rescaled by 2^n since the derivative operator is homogeneous of degree one. The transition matrix R^n has a block tridiagonal structure

$$R^n = \begin{pmatrix} r_0 & r_{-1} & & \\ r_1 & \ddots & \ddots & \\ & \ddots & \ddots & r_{-1} \\ & & r_1 & r_0 \end{pmatrix},$$ (15)

each block r_l being a $\quad\times\quad$ matrix. The matrix blocks r_1 and r_{-1} describe interactions with the left and the right neighboring intervals, respectively, and have rank one as matrices. There are two free parameters associated with the two neighboring intervals, and these define the family of transition matrices. By choosing these parameters, the blocks r_1 or r_{-1} can be made zero, thus providing us with an analogue of forward and backward differences.[3] It is easy to impose a linear boundary condition since such condition amounts to supplying a value to one of the free parameters. The boundary operators have a reasonable condition number since we effectively are using the Gauss-Legendre nodes on the interval (to make it obvious, consider the interpolating scaling functions). The classical Runge example for interpolation then demonstrates the benefits of the Gauss-Legendre nodes versus the equally spaced nodes (which is equivalent to using the usual Multiresolution Analysis and smooth basis functions).

This multiwavelet approach has been used to build an adaptive multiresolution PDE solver for advection-diffusion equations[3] and currently work is under wa

For any $n \geq 0$, the first n functions ψ_j, $j = 0, \ldots, n-1$, form a Chebyshev system.[12,13] In particular, the number of zeros of ψ_j in $[-1, 1]$ is equal to j.

Although the functions ψ_j are defined on the interval, they are easily extended to the whole line using the right hand side of (18) as the definition of the extension. The functions ψ_j are orthogonal on both the interval $[-1, 1]$ and the real line $(-\infty, \infty)$, and we set

$$\int_{-1}^{1} \psi_j(x)\, \psi_l(x)\, dx = \delta_{jl}\,, \tag{21}$$

and

$$\int_{-\infty}^{\infty} \psi_j(x)\, \psi_l(x)\, dx = \frac{1}{\mu_j}\delta_{jl}\,. \tag{22}$$

We note that in the original papers[18,14,17] the functions are chosen to be orthonormal on $(-\infty, \infty)$.

From definition (17) it follows that

$$e^{ic\tilde{}t} = \sum_{j=0}^{\infty} \lambda_j\, \psi_j(x)\psi_j(t)\,. \tag{23}$$

If we keep $\approx 2c/\pi + K \log c$ terms, where $K = K(\epsilon)$ is a constant, we obtain (for any $\epsilon > 0$) an approximation to $e^{ic\tilde{}t}$. This is the most economical expansion of this type for the exponential.

The PSWFs have been used in signal processing for some time, especially the first function, $\psi_0(x)$, since it provides the optimal window for a given bandwidth in terms of concentration in the time-frequency domain. Yet, their use has not been wide. In the next section we describe several new developments that will provide a path for a wider use of these functions in signal processing and numerical analysis.

5. GENERALIZED GAUSSIAN QUADRATURES FOR EXPONENTIALS

The generalized Gaussian quadratures for exponentials has been developed recen

For a given bandlimit $2c > 0$ and accuracy $\epsilon^2 > 0$, we approximate $u(x)$ on the interval $[-1, 1]$ using the sum

$$\tilde{u}(x) = \sum_{k=1}^{M} w_k \exp\left(2c\, t_k\, x\right), \tag{26}$$

where $w_k > 0$ and $M = M(c, \epsilon$

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