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I Introduction

The wavelet bases provide a system of coordinates in which wide classes of linear operators are sparse. As a result, the cost of evaluating Calderon-Zygmund or pseudo-di erential operators on a function is proportional to the number of signi cant wavelet coe cients of this function, i.e., the number of wavelet coe cients above a given threshold of accuracy. Consequently, fast algorithms are now available for solving integral equations with operators from these classes [3].

In order to use the wavelet bases for solving partial di erential equations, one is led to consider di erential operators and operators of multiplication by a function. Numerical issues of representing di erential operators has been addressed

In this paper we address the problem of pointwise multiplication of functions in the wavelet bases. We will consider computing ( ) = ^{2} in the wavelet bases since the product of two functions may be written as = $\frac{1}{4}$[(+)$^2 -$ ($-$)^2].

It appears that the straightforward algorithm which would require computing the expansion of the products of the basis functions, storing and using them to perform the multiplication is ine cient. Such algorithm requires computing the coe cients

$$c^{j;j';m}_{k;k';l} = \int_{-\infty}^{+\infty} {}^{j}_{k}(\ ) \ {}^{j'}_{k'}(\ ) \ {}^{m}_{l}(\ ) d \ M$$

where ${}^{j}_{k}(\ ) = 2^{-j/2} \ (2^{-j} \ - \ )$ are the basis functions. While computing $c^{j;j';m}_{k;k';l}$ does not present a problem, the number of the nonzero of coe cients is large and, what is more important, the number of operations to compute ^{2} is proportional to N_s^3, where N_s is the number of signi cant coe cients in the representation of .

In a number of applications the functions of interest are the functions that are singular or oscillatory at a few locations. The number of signi cant wavelet coe cients of such functions is (1) on each scale so that N_s is proportional to log(

II Multiresolution algorithm for evaluating u^2

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Let us consider the projections of $\in L^2(R)$ on subspaces V_j,

$$_j = \quad _j \; M \quad _j \in V_j M \tag{2.1}$$

where $\{V_j\}_{j\in Z}$,is a multiresolution analysis of $L^2(R)$. In order to uncouple the interaction between scales, we write a \telescopic" series,

$$^2_0 - \; ^2_n = \sum_{j=1}^{j=n} \left[(\; _{j-1} \;)^2 - (\; _j \;)^2 \right] = \sum_{j=1}^{j=n} (\; _{j-1} \; + \; _j \;)(\; _{j-1} \; - \; _j \;) \tag{2.2}$$

Using $_{j-1} = \; _j + \; _j$, we obtain

$$^2_0 - \; ^2_n = \sum_{j=1}^{j=n} (2 \; _j \; + \; _j \;)(\; _j \;) M \tag{2.3}$$

or

$$^2_0 = 2 \sum_{j=1}^{j=n} (\; _j \;)(\; _j \;) +$$

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Let us start by considering an example of (2.4) in the Haar basis. We have the following explicit relations,

$$
\begin{aligned}
({}^{j}_{k}( ))^2 &= 2^{-j/2}\ {}^{j}_{k}( ), \\
({}^{j}_{k}( ))^2 &= 2^{-j/2}\ {}^{j}_{k}( ), \\
{}^{j}_{k}( )\ {}^{j}_{k}( ) &= 2^{-j/2}\ {}^{j}_{k}( ),
\end{aligned}
\tag{2.8}
$$

where ${}^{j}_{k}( ) = 2^{-j/2}\ (2^{-j} -)$, ${}^{j}_{k}( ) = 2^{-j/2}\ (2^{-j} -)$, is the characteristic function of the interval (0, 1) and is the Haar function, $( ) = (2) - (2 - 1)$.

Expanding $_0$ into the Haar basis,

$$
{}_0( ) = \sum_{j=1}^{j=n} \sum_{k\in Z} d^{j}_{k}\ {}^{j}_{k}( ) + \sum_{k\in Z}\ {}^{n}_{k}\ {}^{n}_{k}( ), \tag{2.9}
$$

and using (2.8), x_k

scale = 1, we compute the di erences and averages $d_{\mathbf{k}}^{\mathbf{j}+1}$ and $_{\mathbf{k}}^{\mathbf{j}+1}$. We then add $_{\mathbf{k}}^{\mathbf{j}+1}$ to $\hat{}_{\mathbf{k}}^{\mathbf{j}+1}$ before expanding it further according to the following pyramid scheme

$$\begin{array}{ccccccccc} \{\hat{}_{\mathbf{k}}^{1}\} & \longrightarrow & \{\ _{\mathbf{k}}^{2}\} & \longrightarrow \{\ _{\mathbf{k}}^{2}\} + \{\hat{}_{\mathbf{k}}^{2}\} & \longrightarrow & \{\ _{\mathbf{k}}^{3}\} & \longrightarrow \{\ _{\mathbf{k}}^{3}\} + \{\hat{}_{\mathbf{k}}^{3}\} & \cdots & \\ & \searrow & & & \searrow & & & \searrow & \\ & & \{d_{\mathbf{k}}^{2}\} & \longrightarrow \{d_{\mathbf{k}}^{2}\} + \{\hat{d}_{\mathbf{k}}^{2}\} & & \{d_{\mathbf{k}}^{3}\} & \longrightarrow \{d_{\mathbf{k}}^{3}\} + \{\hat{d}_{\mathbf{k}}^{3}\} & \cdots & \end{array} \quad (2.13)$$

(The formulas for evaluating the di erences and averages $d_{\mathbf{k}}^{\mathbf{j}+1}$ and $_{\mathbf{k}}^{\mathbf{j}+1}$ may be found in [3]). As a result, we compute $d_{\mathbf{k}}^{\mathbf{j}}$, = 2$M$ M, (we set $d_{\mathbf{k}}^{1}$ = 0) and $_{\mathbf{k}}^{\mathbf{n}}$ and obtain

$$_0^2(\ ) = \sum_{\mathbf{j}=1}^{\mathbf{j}=\mathbf{n}} \sum_{\mathbf{k}\in\mathbf{Z}} (\hat{d}_{\mathbf{k}}^{\mathbf{j}} + d_{\mathbf{k}}^{\mathbf{j}}) \ _{\mathbf{k}}^{\mathbf{j}}(\ ) + \sum_{\mathbf{k}\in\mathbf{Z}} (\ _{\mathbf{k}}^{\mathbf{n}} + \hat{}_{\mathbf{k}}^{\mathbf{n}} + \hat{}_{\mathbf{k}}^{\mathbf{n}}) \ _{\mathbf{k}}^{\mathbf{n}}(\ ) \quad (2.14)$$

It is clear, that the number of operations for computing the Haar expansion of $_0^2$ is proportional to the number of signi cant coe cients $d_{\mathbf{k}}^{\mathbf{j}}$ in the wavelet expansion of $_0$. In the worst case, if the original function is represented by a vector of the length N, then the number of operations is proportional to N. If the original function is represented by (log$_2 N$) signi cant Haar coe cients, then the number of operations to compute its square is proportional to log$_2 N$. The algorithm in the Haar basis easily generalizes to the multidimensional case.

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We now return to the general case of wavelets and derive an algorithm to expand (2.4) into the wavelet bases. Unlike in the case of the Haar basis, the product on a given scale "spills over" into the ner scales and we develop an e cient approach to handle this problem. We use compactly supported wavelets though our considerations are not restricted to such wavelets. We denote the scaling function by and the wavelet by . The wavelet basis is then given by $_{\mathbf{k}}^{\mathbf{j}}(\ ) = 2^{-\mathbf{j}/2} \ (2^{-\mathbf{j}} \ - \ )$, $M \in \mathbb{Z}$ (see [8]). We consider the multiresolution analysis associated with such basis.

In order to expand each term in (2.4) into the wavelet basis we are led to consider the integrals of the products of the basis functions, for example

$$M_{\mathbf{www}}^{\mathbf{j};\mathbf{j}'}(\ M' M) = \int_{-\infty}^{+\infty} \ _{\mathbf{k}}^{\mathbf{j}}(\ ) \ _{\mathbf{k}'}^{\mathbf{j}}(\ ) \ _{\mathbf{l}}^{\mathbf{j}'}(\ ) \, d \ M \quad (2.15)$$

where $' \leq$. It is clear, that the coe cients $M_{\mathbf{www}}^{\mathbf{j};\mathbf{j}'}(\ M' M)$ are identically zero for $| \ - \ '| > \ _0$, where $_0$ depends on the overlap of the supports of the basis functions. The number of necessary coe cients may be reduced further by observing that

$$M_{\mathbf{www}}^{\mathbf{j};\mathbf{j}'}(\ M' M) = 2^{-\mathbf{j}'/2} \int_{-\infty}^{+\infty} \ _0^{\mathbf{j}-\mathbf{j}'}(\ ) \ _{\mathbf{k}-\mathbf{k}'}^{\mathbf{j}-\mathbf{j}'}(\ ) \ _{2^j \ j'\mathbf{k}-\mathbf{l}}^{0}(\ ) \, d \ M \quad (2.16)$$

Though it is a simple matter to derive and solve a system of linear equations to nd M_0(, M), we advocate a di erent approach to evaluate (2.24) in the next subsection.

Let us now explain the reasons for considering (2.20) and (2.21) as mappings (2.24). On a given scale the procedure of "lifting" the projections $_j$, $_j$ into a " ner" subspace is accomplished by the pyramid reconstruction algorithm (see e.g. [3]). Let us assume that only a small number of the coe cients of $_j$ are above the threshold of accuracy. We note (see Remark 2 for the Haar basis) that only those coe cients of $_j$ that contribute to the product ($_j$)($_j$) (above the threshold) need to be kept. In fact, one may consider the function

Instead of (2.24), it is su cient to consider the mapping

$$\mathrm{V}_0 \times \mathrm{V}_0 \to \mathrm{V}_0 \qquad (2.27)$$

It is easy to see that for $\in \mathrm{V}_0$,

$$(\ ) = \sum_{k} {}_{k} (\ - \ ) M \qquad (2.28)$$

the values of at integer points may be wri-a9.35955 0 Td95 0 Td (ts)Tj 13891 0 Td (b)Tj 6.83967as

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