DECONVOLUTION AND INVERSION

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MATHEMATICAL THEORY FOR SEISMIC

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INTRODUCTION

Migration methods in seismics are computationally intensive and powerful tools for interpretation of seismic experiments. A partial list of papers on migration methods in the geophysical literature include those by Hagedoorn (1954), Lindsey and Herman (1970), Rockwell (1971), Claerbout (1971), Schneider (1971, 1978), Claerbout and Doherty (1972), French (1974, 1975), Gardner *et al.* (1974), Cohen and Bleistein (1977, 1979), Stolt (1978), Berkhout (1980, 1984), Clayton and Stolt (1981), Johnson and French (1982), Devaney (1984), Gazdag and Squazzero (1984), Tarantola (1984), Bleistein and Gray (1985), Stolt and Weglein (1985) and many more. This list derivation of migration algorithms to solve the linearized inverse scattering problem is best understood within the theory of pseudodifferential and Fourier integral operators.

The sequence for the derivation of a migration algorithm within this theory can be briefly described as follows. For a given background

17.1 Pseudodifferential and Fourier Integral Operators

The title of this section might discourage a non-mathematician since these notions are not yet a part of the standard curriculum in applied mathematics. However, these mathematical objects are well known under different

A more symmetric form of the symbol with

The Sobolev space $H^s(R^d)$ is the space of distribution in R^d whose Fourier transform is a square-integrable function in R^d with the measure $(1+|k|^2)^s\,dk$. It is a Hilbert space with the inner product

$$(u, v) = \frac{1}{(2\pi)^d} \int_{R^d} \hat{u}(k)\bar{\hat{v}}(k)(1+|k|^2)^s\,dk, \tag{17.11}$$

where the bar denotes complex conjugation.

The subspace $H^s_{comp}(Q)$ of $H^s(R^d)$ consists of the distributions with the support in the compact set Q; $H^s_{comp}(X)$ is the union of the spaces $H^s_{comp}(Q)$, where Q spans the collection of all compact subsets of X. Finally, $H^s_{loc}(X)$ is a space of distributions in X, such that if properly localized (by infinitely differentiable cutoff functions to compact subsets exhausting X) those operator $\mathbf{P} \in L^m$ an asymptotic expansion of this operator can be constructed:

$$\mathbf{P} = \mathbf{T}_m + \mathbf{T}_{m-1} + \mathbf{T}_{m-2} + \dots, \tag{17.13}$$

where

$$\mathbf{T}_j \in L^j(X), \tag{17.14}$$

for $j = m, m-1, m-2, \dots$, and

$$(\mathbf{P} - \mathbf{T}_m - \mathbf{T}_{m-1} - \cdots - \mathbf{T}_j) \in L^{j-1}(X), \tag{17.15}$$

for $j = m, m-1, m-2, \dots$. Such an expansion is modulo regularizing operators

Fourier integral operators

An operator of the form

$$(F f)(x) = \frac{1}{\quad} \int \int f(y) A(x, y, k) e^{i\Phi(x, y, k)} \, dy \, dk \tag{17.16}$$

Given the background index of refraction n_0 (background model) the linearized

inverse scattering problem is that of characterization of the perturbation f using observations of the scattered field u on the boundary ∂X of the region X.

If the propagation is governed by the Helmholtz equation, then the scattered field $\hat{u}(s, r, \omega)$ satisfies within the single scattering (or distorted wave Born) approximation the following integral equation

$$\hat{u}(s, r, \omega) = -\omega^2 \int_X G(y, r, \omega) f(y) G(s, y, \omega)\, \mathrm{d}y, \tag{17.19}$$

where G is the Green's function of the background model.

The Green's function G is the solution of the equation

$$(\nabla_y^2 + \omega^2 n_0^2(y)) G(y, r, \omega) = \delta(y - r), \tag{17.20}$$

and, in principle, can be computed given the background model. The incident field $G(s, y, \omega)$ is due to the point source located at the point s.

Remark 1. Implicit in (17.20) is the assumption that the boundary ∂X is not physical (in our case it means that the index of refraction does not have a jump at ∂X). If the boundary is a physical boundary then the definition of the Green's function changes and includes boundary conditions. In what follows it only affects the computation of

amplitudes.

Remark 2. Equation (17.19) is obtained via standard perturbation analysis. The range of validity of this approximation was discussed by a number of authors and is not treated here.

We view (17.19) as an equation for the unknown function f. This is an integral equation with an oscillatory kernel. We proceed to solve this equation by constructing a Fourier integral operator (which can be shown to be a pseudodifferential operator) and to compute the first term of its asymptotics.

We start by noting that the simplest integral equation with an oscillatory kernel is the Fourier transform

$$\hat{f}(k) = \int_X f(y) \mathrm{e}^{-ik\cdot y}\, \mathrm{d}y. \tag{17.21}$$

The solution of this equation is obtained by applying the adjoint operator (the inverse Fourier transform),

$$f(x) = \frac{1}{(2\pi)^3} \int_{R^3} \hat{f}(k) \mathrm{e}^{ik\cdot x}\, \mathrm{d}k. \tag{17.22}$$

where the function $h(s, r, x)$ is yet to be described. Here Re denotes the real part of the where

$$\Phi(s, r, x) = \phi(s, x) + \phi(x, r), \tag{17.29}$$

is the total travel time between the source, the point x, and the receiver.

Step 2. At this step we localize the computation to the neighbourhood of the point of reconstruction x. If $\varepsilon < |x-y|$, where ε is any positive number, then the result of integration in (17.28) (over the part of the domain X described by this condition) is infinitely differentiable and, therefore, will not affect the asymptotics. If $|x-y| < \varepsilon$ we replace the phase of the exponent by the first term of the Taylor series

$$\Phi(s, r, x) - \Phi(s, r, y) = \nabla_x \Phi(s, r, x) \cdot (x-y), \tag{17.30}$$

and 'freeze' the value of the amplitude terms at the point x. By doing this we account for the most singular term in the asymptotic expansion with respect to smoothness. We obtain from (17.13)

$$f_{..}(x) = \frac{1}{\ldots} \mathrm{Re} \int^{\infty} \int \int e^{i\omega \nabla_x \Phi(s, r, x) \cdot (x-y)} \ldots$$ [illegible]

Step 3. At this step we set $\omega^2 h(s, r, x)$ to be the Jacobian of the change of variables from $\omega \in [0, \infty]$ and $r \in \partial X_r$ to $k \in R^3$. We have

$$k = \omega \nabla_x \Phi(s, r, x), \tag{17.32}$$

so that

$$\mathrm{d}k = h(s, r, x) \mathrm{d}r \omega^2 \mathrm{d}\omega. \tag{17.33}$$

The function $h(s, r, x)$ can be computed by ray tracing using the identity (Beylkin 1985a)

$$h(s, r, x) \mathrm{d}r = n_0^3 (1 + \cos \psi(s, r, x)) \mathrm{d}\Omega, \tag{17.34}$$

where

$$\cos \psi(s, r, x) = \frac{\nabla_x \phi(s, x) \cdot \nabla_x \phi(x, r)}{\ldots}$$ [illegible]

coverage in the space of spatial frequencies. This domain is determined by the map we have

$$f_{est}(s) = -\frac{1}{\ldots}\int \frac{h(s, r, x)}{\ldots}$$

Concluding remarks

This paper demonstrates (without proofs) how to derive migration algorithms using tools from the theory of pseudodifferential and Fourier integral operators. The purpose of this presentation is to discuss the mathematical technique as it is applied in the context of seismic problems rather than to propose a specific algorithm. For this reason, instead of giving numerical examples, I refer to the papers Miller *et. al.* (1984, 1987), and Beylkin *et. al.* (1985), where specific algorithms are presented along with

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