

On Generalized Gaussian Quadratures for Exponentials and Their Applications

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We introduce new families of Gaussian-type quadrature formulas for weighted integrals of exponential functions and consider their application to integration and interpolation of bandlimited functions.

We give a generalization of a representation theorem due to Caithodoro to derive the quadrature. For each positive measure, the quadrature is a parameterized eigenvalue of the Toeplitz matrix, constructed from the trigonometric moments of the measure. For a given accuracy

The new quadrature can be used to approximate and integrate bandlimited functions, such as prolate spheroidal wave functions, and essentially bandlimited functions, such as Beĭlkin functions. We also develop, for a given precision, an interpolating basis for bandlimited functions on an interval.

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1. INTRODUCTION

In this paper we relate the Caithodoro representation of finite sequences in terms of exponential functions with the computation of generalized Gaussian quadrature. Keĭn and Nadel'man [13], Kalin and Studden [12] and, more recently, Bayin and Rokhlin [30]. In [30] the authors introduce practical algorithms for computing the nodes and weights of generalized Gaussian quadrature. The resulting approximation has a number of important applications in a variety of fast algorithms [3, 31].

the form

$$c_k = \sum_{j=1}^M j e^{i j k}, \quad (1.1)$$

for $k = 1, 2, \dots, N$ and $M \geq N$, where $-1 < j < 1$ and $j > 0$. Cauchy's representation (1.1) has been the foundation for a number of algorithms for spectral estimation; in particular, [20] is known in electrical engineering literature as the Pisarenko method. In this paper we develop a fast algorithm for finding M , the phase T015

A method for constructing generalized Gaussian quadrature, or else limited to integral (with a fairly arbitrary measure) in solving exponential. Orthogonalizing eigenvalue and eigenvectors of a Toeplitz matrix, constructed from trigonometric moments of the measure and then computing the roots on the unit circle for appropriate eigenpolynomial. In particular, each eigenpolynomial with distinct roots give rise to an identity which, for small eigenvalue, provide with a Gaussian-type quadrature and also with a representation of positive definite Hermitian Toeplitz matrices. In the identities the size of the eigenvalue determine the accuracy of the quadrature formula.

It is noted that in the case of the eight leading to PSWF, the nodes of the corresponding Gaussian quadrature are zero (appropriately scaled to the interval $[-1, 1]$) of discrete PSWF corresponding to small eigenvalue.

As an application, we give the new quadrature 7(I)-4.9(n)-199.2gTj/Fx.7ng2(cited)cTj4o2a . 4.7

The paper is organized as follows. We present a brief description of the Piavko method to obtain the classical Carathéodory representation and we describe the estimate (1.2) in Section 2. In Section 3 we discuss generalized Gaussian quadrature for weighted integrals and prove some of their properties for weight supported in $[-1/2, 1/2]$. In Section 4 we introduce new families of Gaussian-type quadrature. We develop a fast algorithm in Section 5 to compute the nodes and weights of the quadrature. We solve the approximation problem (1.3)–(1.5) in Section 6 and describe it in the next section to obtain quadrature and interpolating bases for bandlimited functions. We also discuss a numerical example to illustrate the results. Finally, conclusions are presented in Section 9.

2. CARATHÉODORY REPRESENTATION

Carathéodory representation of the trigonometric moment problem and can be stated as follows (see [8, Chap. 4]).

THEOREM 2.1. *Given N complex numbers $\mathbf{c} = (c_1, c_2, \dots, c_N)$, not all zero, there exist unique $M \leq N$, positive numbers $\boldsymbol{\rho} = (\rho_1, \rho_2, \dots, \rho_M)$, and distinct real numbers $\theta_1, \theta_2, \dots, \theta_M$*

2.1. Algorithm 1: Method to Obtain M , θ , and ρ

(1) Given $c = (c_1, c_2, \dots, c_N)$, extend the definition of c_k to negative k as $c_{-k} = \overline{c_k}$ and define c_0 so that the $(N + 1) \times (N + 1)$ Toeplitz matrix T_N of elements $(T_N)_{kj} = c_{j-k}$, has nonnegative eigenvalues and at least one eigenvalue equal to zero.

(2) Define M as the rank of T_N . By construction, $M \leq N$. We also assume that M is the rank of the representation (2.1).

(3) Let T_M be the top left principal submatrix of order $M + 1$ of T_N . That is, the matrix T_M has elements $(c_{j-k})_{0 \leq k, j \leq M}$. Find the eigenvector q corresponding to the zero eigenvalue of T_M .

(4) Construct the polynomial (eigenpolynomial) whose coefficients are the entries of the eigenvector q . According to [8, p. 58], the M roots of this eigenpolynomial are distinct and have absolute value 1. The phase of the roots are the numbers θ_j .

(5) Find the eigenvector ρ by solving the Vandermonde system (2.1) for $k = 1, \dots, M$. The vector, in addition, satisfies $\sum_k \rho_k = c_0$.

Remark 2.1. With the extension of the sequence c_k , (2.1) is valid for $|k| \leq N$. If $q = (q_0, \dots, q_M)$ is the eigenvector obtained in part (3) of Algorithm 2.1, then

$$\sum_{k=0}^M c_{k+s} q_k = 0, \tag{2.2}$$

for all $s, -N \leq s \leq 0$. In other words, we have found an order- M recurrence relation for the original sequence $\{c_k\}_{k=1}^N$.

Remark 2.2. In practice, we are interested in finding a stable representation if M is small compared with N , or more generally, if most eigenvectors are smaller than the accuracy. However, in such cases, T_N has a large (numerical) null subspace that causes a numerical problem in determining c_0 , the rank M , and the eigenvector q .

Nevertheless, if the sequence c is the trigonometric moment of an appropriate eigenvector, we will be able to modify the procedure in order to obtain the phases θ_j in an efficient manner. In this setting, the phases and eigenvectors in a stable representation can be thought of as the nodes and eigenvectors of a Gaussian-type quadrature for weighted integrals. Once the phases are obtained, Theorem 2.2 assures that the computation of the eigenvectors is a well-posed problem. In Section 5.2, we present a fast algorithm to obtain the eigenvectors by evaluating certain polynomials at the nodes $e^{i\theta_j}$.

Remark 2.3. Given an Hermitian Toeplitz matrix T , let us consider it as a matrix of eigenvalues (N)

3. GENERALIZED GAUSSIAN QUADRATURES FOR EXPONENTIALS

3.1. Preliminaries: Chebyshev Systems

In this section we collect some definitions and results related to Chebyshev systems. We follow mostly Karlin and Studden [12] (see also [13]). Readers familiar with this topic may skip this section.

A family of $n + 1$ real-valued functions u_0, \dots, u_n defined on an interval $I = [a, b]$ is a Chebyshev system (T-system) if an nontrivial linear combination

$$u(t) = \sum_{j=0}^n \alpha_j u_j(t) \tag{3.1}$$

has at most n zeros on the interval I . This property of a T-system can be viewed as a generalization of the same property for polynomials. Indeed, the family $\{1, t, t^2, \dots, t^n\}$ provides the simplest example of a Chebyshev system.

Alternatively, a T-system on $[a, b]$ may be defined by the condition that the $n + 1$ order determinant is nonvanishing,

$$\det \begin{pmatrix} u_0(t_0) & u_0(t_1) & \dots & u_0(t_n) \\ u_1(t_0) & u_1(t_1) & \dots & u_1(t_n) \\ \dots & \dots & \dots & \dots \\ u_n(t_0) & u_n(t_1) & \dots & u_n(t_n) \end{pmatrix} = 0, \tag{3.2}$$

where $a < t_0 < t_1 < \dots < t_n < b$. Without loss of generality, the determinant can be assumed positive.

Let u_0, \dots, u_n be a T-system on the interval I . The moment space \mathcal{M}_{n+1} with respect to u_0, \dots, u_n is defined as the set

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$$\mathcal{M}_{n+1} =$$

THEOREM 3.5 [12, VI, Sec. 4]. *For the periodic T -system (3.5), a point*

In this section we tabling Cauchy's theorem and Theorem 3.7 from the previous section, to construct ν -different Gaussian quadrature formulae for integral ν -th weight w . The quadrature are exact for trigonometric polynomial of appropriate degree.

We then generalize the type of quadrature formulae and develop a new family of Gaussian-type quadrature. This family of quadrature formulae is parameterized by the eigenvalue of the Toeplitz matrix.

$$T = \{t_{l-k}\}_{0 \leq k, l \leq N}. \tag{4.2}$$

Among the new quadrature formulae, only those corresponding to eigenvalue of small magnitude are of practical interest. In fact, the size of the eigenvalue determine the error of the quadrature formula. To compute the ν -weight and node of the quadrature, we develop a new algorithm which may be considered as a (major) modification of Algorithm 2.1. The new algorithm is described in Section 5. The main result of this section are gathered in Theorem 4.1.

We tabling Theorem 3.7 to write

$$t_k = \sum_{j=1}^N j e^{i \theta_j k} + c_0 (-1)^k, \quad \text{for } |k| \leq N, \tag{4.3}$$

for unique positive ν -weight j and phase θ_j in $(-1, 1)$. Then, for an $A(z) = \sum_{|k| \leq N} a_k z^k$ in N , the space of Laurent polynomial of degree at most N , we have

$$\int_{-1}^1 A(e^{i \theta}) w(\theta) d\theta = \sum_{|k| \leq N} a_k t_k = \sum_{j=1}^N j A(e^{i \theta_j}) + c_0 A(-1), \tag{4.4}$$

for unique positive ν -weight j and node $e^{i \theta_j}$.

Alternatively, using Cauchy's theorem (2.1) applied to the sequence $c_k = t_k$, $1 \leq k \leq N$,

$$\begin{aligned} \int_{-1}^1 A(e^{i \theta}) w(\theta) d\theta &= \sum_{j=1}^M j A(e^{i \theta_j}) + (t_0 - c_0) \frac{1}{2} \int_{-1}^1 A(e^{i \theta}) d\theta \\ &= \sum_{j=1}^M j A(e^{i \theta_j}) + c_0 \frac{1}{2} \int_{-1}^1 A(e^{i \theta}) d\theta, \end{aligned} \tag{4.5}$$

where $c_0 = \sum_{j=1}^M j$ and $\{e^{i \theta_j}\}$ are the roots of the eigenpolynomial corresponding to the smallest eigenvalue c_0 of T .

Note that (4.5) is again valid for all $A(z)$ in N and that the positive ν -weight j and phase θ_j in $(-1, 1]$ are unique.

The ν -weight ν -different quadrature that may not coincide. However, by considering $w(\theta)$ proposed in $(-1/2, 1/2)$, (3.12) implies that w_0 in (4.4) decays exponentially fast with N and, since $\min w(\theta) = 0$ for $|\theta| \leq 1$, we have

$$\lim_{N \rightarrow \infty} c_0^{(N)} = 0, \tag{4.6}$$

4.2. Gaussian-Type Quadratures on the Unit Circle

In this section we present the main result of the paper. We define the Gaussian-type quadrature valid for an eigenvalue of the matrix T other than just the smallest eigenvalue λ_N . The quadrature allows us to select the desired accuracy and then, to construct accuracy-dependent families of quadratures.

The nodes of the quadrature in (4.5) are the roots of the eigenpolynomial corresponding to the least eigenvalue of T and, because of Cauchy's interlacing theorem, we know that the nodes are on the unit circle and that the weights are positive numbers. In our generalization, this standard property for the nodes and weights is no longer enforced. However, we will show that for nodes on the unit circle, the corresponding weights are real. Moreover, in all examples we have examined, for all small eigenvalues of T , the negative weights are associated with the nodes outside the support of the weights and are comparable in size with them. We believe this property to hold for a wide class of weights.

We propose the following

THEOREM 4.1. *Assume that the eigenpolynomial $V^{(s)}(z)$ corresponding to the eigenvalue $\lambda^{(s)}$ of T has distinct, nonzero roots $\{z_j\}_{j=1}^N$. Then there exist numbers $\{w_j\}_{j=1}^N$ such that*

(i) *For all Laurent polynomials $P(z)$ of degree at most N ,*

$$\int_{-1}^1 P(e^{it})w(t) dt = \sum_{j=1}^N w_j P(z_j) + \lambda^{(s)} \frac{1}{2} \int_{-1}^1 P(e^{it}) dt. \tag{4.12}$$

(ii) *For each root z_k with $|z_k| = 1$, the corresponding weight w_k is a real number and*

$$w_k = \int_{-1}^1 |L_k^s(e^{it})|^2 w(t) dt - \lambda^{(s)} \frac{1}{2} \int_{-1}^1 |L_k^s(e^{it})|^2 dt, \tag{4.13}$$

where

$$L_k^s(z) = \frac{V^{(s)}(z)}{(V^{(s)}(z_k)(z - z_k))} \tag{4.14}$$

is the Lagrange polynomial associated with the root z_k .

(iii) *If $\lambda^{(s)}$ is a simple eigenvalue, then for $k = 1, \dots, N$, the weight w_k is nonzero and*

$$\frac{1}{w_k} = \sum_{l=s}^N \frac{V^{(l)}(z_k) V^{(l)}(z_k)}{(V^{(l)}(z_k) - V^{(s)}(z_k))}, \tag{4.15}$$

where $V^{(l)}(z) = \overline{V^{(l)}(z^{-1})}$ is the reciprocal polynomial of $V^{(l)}(z)$.

In particular, for each z_k with $|z_k| = 1$,

$$\frac{1}{w_k} = \sum_{l=s}^N \frac{|V^{(l)}(z_k)|^2}{(V^{(l)}(z_k) - V^{(s)}(z_k))}. \tag{4.16}$$

(i) *If $\lambda^{(s)}$ is a simple eigenvalue and all roots z_k are on the unit circle, then the set $\{w_k\}_{k=1}^N$ contains exactly s positive numbers and $N - s$ negative numbers.*

In particular, if $s = 0$ or $s = N$, then all w_k are negative or positive, respectively.

Remark 4.1. Our approach to obtain Gaussian quadrature does not use Stieltjes polynomials and is therefore substantially different than the one in [11]. We briefly explain the approach in [11]. Note that (4.9) and (4.10) show that the polynomials $\{V^{(k)}(z)\}$ are orthogonal with respect to both the usual inner product for trigonometric polynomials and the weighted inner product with weight $w(t)$. We can also construct Stieltjes polynomials $\{p_k(z)\}$ orthogonal with respect to $w(t)$ and check that each $p_k(z)$ has precisely k [26]. For an k , the roots of $p_k(z)$ are all in $|z| < 1$ [8].

Stieltjes polynomials and their reciprocal induce para-orthogonal polynomials [11],

$$p \quad e \quad r \quad a B_n B$$

For k outside the support of the measure, v_k has been observed (Fig. 2, 3, and 5.8) that

$$\sum_{l: \binom{s}{l} > 0} |V^{(l)}(z_k)|^2$$

is a constant of moderate size.

Thus, the second term in (4.17) is $O(1/\binom{s}{l})$ and the v_k might indeed be negative and orthogonal to the eigenspace.

Remark 4.5. For the v_k with z_k all real in $(-1/2, 1/2)$ and 0 otherwise, the eigenpolynomial is the discrete PSWF. For the definition, v_k are known that all eigenvalues are simple and that all eigenpolynomials are on the unit circle [23].

COROLLARY 4.1. *Under the assumptions of Theorem 4.1, it follows that the Toeplitz matrix T in (4.2) has the following representation as a sum of rank-1 Toeplitz matrices,*

$$(T - \binom{s}{l} I)_{kl} = \sum_{j=1}^N w_j z_j^{l-k},$$

where $\binom{s}{l}$, w_j , and z_j are as in (4.12).

This corollary should be compared with Remark 2.3 noting that, in the corollary, $\binom{s}{l}$ is not necessarily the least eigenvalue of T . For an alternative definition see [4].

Proof of Theorem 4.1. (1) For $x = (x_0, \dots, x_N) \in \mathbb{C}^{N+1}$, let us define

$$A_x(z) = \sum_{l=-L}^L x_{l+L} z^l, \quad \text{if } N = 2L$$

<

(3) Let $P \in \mathbb{C}[z]$; then $z^N P(z)$ is a polynomial of at most degree $2N$, and since $z^L V^{(s)}(z)$ is a polynomial of degree N , by Euclidean division, there exist polynomials $q(z)$ and $r(z)$ of degree at most N and $N - 1$ such that

$$z^N P(z) = z^L V^{(s)}(z)q(z) + r(z).$$

Then,

$$P(z) = V^{(s)}(z)Q(z) + R(z), \tag{4.19}$$

where $Q(z) \in \mathbb{C}[z]$ and $R(z)$ has the form $R(z) = \sum_{k=1}^N r_k z^{-k}$ and hence

$$\int_{-1}^1 R(e^{i t}) dt = 0.$$

Using the fact that $\{V^{(l)}\}_{l=0}^N$ is a basis of $L_{\mathbb{C}} e_{\mathbb{C}}$ it is

$$\overline{Q(e^{i t})} = \sum_{l=0}^N d_l V^{(l)}(e^{i t}),$$

where d_l are some complex coefficients.

Using (4.10) and (4.18), we multiply both sides of (4.19) by $w(t)$ and integrate to obtain

$$\int_{-1}^1 P(e^{i t})w(t) dt = \sum_{l=0}^N d_l \int_{-1}^1 V^{(l)}(e^{i t})w(t) dt$$

and then, considering $k = j$, (4.15) follows. Note that we need $V^{(s)}$ to be simple to guarantee $V^{(l)} - V^{(s)} = 0, l = s$ in (4.20).

If we use the left hand side of (4.20) as the entries A_{kj} of a matrix A and let B be the matrix of entries

$$B_{lk} = V^{(l)}(x_k), \quad \text{where } 0 \leq l \leq N, l = s, \text{ and } 1 \leq k \leq N, \quad (4.21)$$

we can prove (4.20) by showing that $BA = B$ and that B is nonsingular.

For the latter claim, we simply check that the columns of B are linearly independent. Indeed, let $a_l, l = s$, be constants such that

$$\sum_{l=s} a_l V^{(l)}(x_k) = 0, \quad \text{for } k = 1, \dots, N.$$

It follows that the polynomial $P(z) = \sum_{l=s} a_l V^{(l)}(z)$ of degree L has the $N = 2L$ distinct roots x_k . Since P and $V^{(s)}$ have the same degree and the same N distinct roots, $P(z) = cV^{(s)}(z)$, for some constant c . By (4.9), $V^{(s)}(z)$ is orthogonal to all the other eigenpolynomials and so $a_l = 0$.

To show that $BA = B$, we substitute $P(z) = V^{(l)}(z)V^{(m)}(z)$ in (4.12) to obtain

$$\int_{-1}^1 V^{(l)}(e^{it}) \dots$$

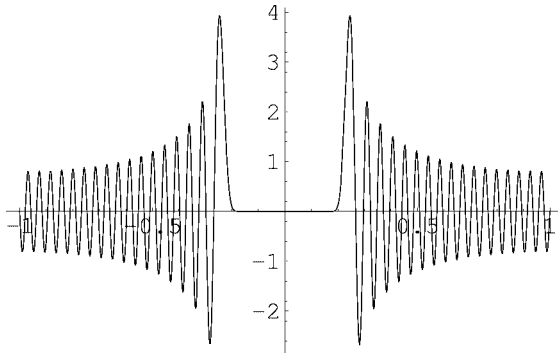


FIG. 2. Modified eigenpolynomial $e^{-i t(N/2)}V^{(30)}(e^{i t})$ on the interval $[-1, 1]$, where $N = 97$ and $V^{(30)}(e^{i t})$ is the eigenpolynomial corresponding to the eigenvalue $\lambda^{(30)}$ in Example 1. The phase factor $e^{-i tN/2}$ is introduced to make this function real.

EXAMPLE 1. Fix ν , and consider the ν -eight

$$w(t) = \begin{cases} 1, & t \in [-a, a], \quad a = 1/2, \\ 0, & \text{elsewhere.} \end{cases} \tag{4.24}$$

For this ν -eight, the eigenpolynomial $V^{(l)}(e^{i t})$ of the $(N + 1) \times (N + 1)$ Toeplitz matrix \mathbf{T} are the discrete PSWF [23]. Then the eigenpolynomial $V^{(l)}(e^{i t})$ has all of its zeros on the unit circle. Moreover, it has exactly l zeros for t in the interval $(-a, a)$ and $N - l$ zeros for t in $[-1, 1]$. In this example, we have selected $N = 97$, $a = 1/6$, $c = 15$. We then construct the matrix \mathbf{T} and compute the eigenpolynomial corresponding to the eigenvalue

$$\lambda^{(30)} = 9.77306136381891632828 \cdot 10^{-16}. \tag{4.25}$$

The eigenpolynomial $V^{(30)}(e^{i t})$ is shown in Fig. 2 and 3. Location of the zeros on the unit circle are displayed in Fig. 4. We then determine the quadrature formula corresponding to this eigenvalue and tabulate the ν -eight in Table I. Note that the ν -eight for node in the interval $[-1/6, 1/6]$

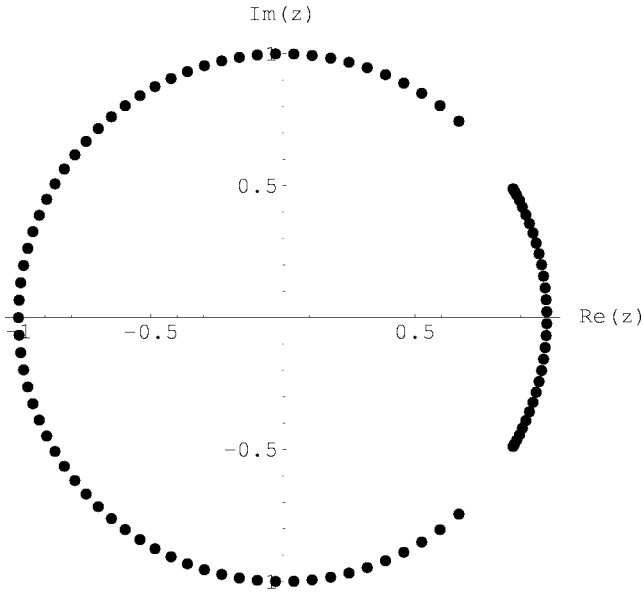


FIG. 4. Location of the e o on the unit circle for the eigenpolynomial $V^{(30)}$ in Example 1.

TABLE I
Table of Weights for the Quadrature Formula with $\lambda^{(30)}$ in Example 1

| # | Weight | # | Weight |
|----|--------------------------|----|--------------------------|
| 1 | $-1.0328 \cdot 10^{-17}$ | 50 | 0.04437549133235668283 |
| 2 | $-1.0328 \cdot 10^{-17}$ | 51 | 0.04419611220330997984 |
| 3 | $-1.0329 \cdot 10^{-17}$ | 52 | 0.04382960375644760677 |
| ⋮ | ⋮ | 53 | 0.04325984471286061543 |
| 33 | $-1.3518 \cdot 10^{-17}$ | 54 | 0.04246105337417774134 |
| 34 | $-1.6030 \cdot 10^{-17}$ | 55 | 0.04139574827622469674 |
| 35 | 0.00580295532842819966 | 56 | 0.04001188663952018400 |
| 36 | 0.01310603337477264417 | 57 | 0.03823923547752508920 |
| 37 | 0.01959211245475268191 | 58 | 0.03598544514201341779 |
| 38 | 0.02506789313597245367 | 59 | 0.03313334531810570720 |
| 39 | 0.02954323947353217723 | 60 | 0.02954323947353217723 |
| 40 | 0.03313334531810570720 | 61 | 0.02506789313597245367 |
| 41 | 0.03598544514201341779 | 62 | 0.01959211245475268191 |
| 42 | 0.03823923547752508920 | 63 | 0.01310603337477264417 |
| 43 | 0.04001188663952018400 | 64 | 0.00580295532842819966 |
| 44 | 0.04139574827622469674 | 65 | $-1.6030 \cdot 10^{-17}$ |
| 45 | 0.04246105337417774134 | 66 | $-1.3518 \cdot 10^{-17}$ |
| 46 | 0.04325984471286061543 | | |
| 47 | 0.04382960375644760677 | | |
| 48 | 0.04419611220330997984 | | |
| 49 | 0.04437549133235668283 | | |

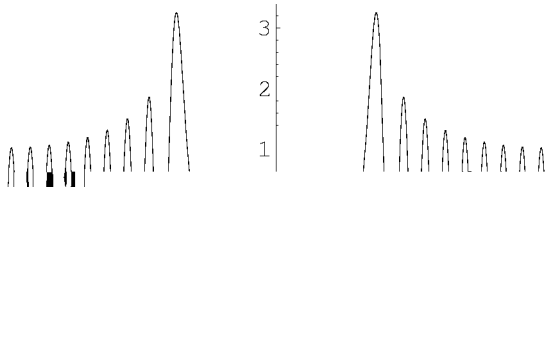


FIG. 5. Modified eigenpolynomial (see Fig. 2) on the interval $[-1, 1]$ corresponding to the eigenvalue $\lambda^{(28)}$ in Example 2.

EXAMPLE 2. We consider the weight

$$w(t) = \begin{cases} |t|/a, & t \in [-a, a], \quad a = 1/2, \\ 0, & \text{elsewhere.} \end{cases} \quad (4.26)$$

In this example we have selected $N = 61$, $a = 1/4$, $c = 15$. We then construct the matrix T and compute the eigenpolynomial corresponding to the eigenvalue

$$\lambda^{(28)} = 1.11598931688523706280 \cdot 10^{-14}. \quad (4.27)$$

The eigenpolynomial $V^{(28)}(e^{it})$ is shown in Fig. 5 and 6.

EXAMPLE 3. We consider a non-symmetric weight

$$w(t) = \begin{cases} 1 + t/a, & t \in [-a, a], \quad a = 1/2, \\ 0, & \text{elsewhere.} \end{cases} \quad (4.28)$$

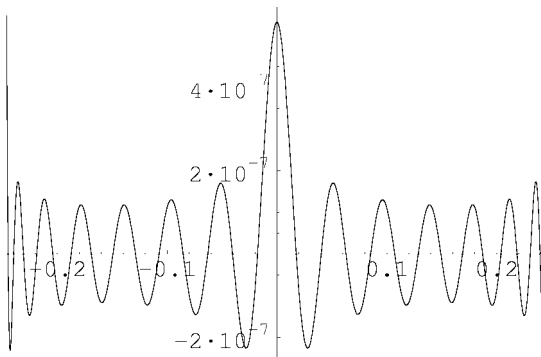


FIG. 6. The same function of Fig. 5 on the interval $[-1/4, 1/4]$.

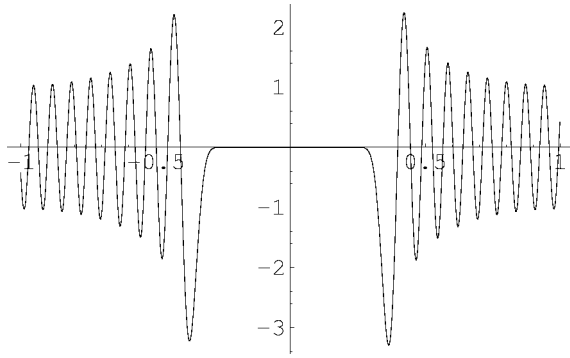


FIG. 7. Modified eigenpolynomial (see Fig. 2) on the interval $[-1, 1]$ corresponding to the eigenvalue $\lambda^{(28)}$ in Example 3.

In this example we have selected $N = 61$, $a = 1/4$, $c = 15$. We then construct the matrix T and compute the eigenpolynomial corresponding to the eigenvalue

$$\lambda^{(28)} = 4.68165338379692121389 \cdot 10^{-15}. \tag{4.29}$$

The eigenpolynomial $V^{(28)}(e^{it})$ is shown in Fig. 7 and 8. Although we do not have a proof at the moment, it appears that the eigenvalues of eight different eigenpolynomials corresponding to small eigenvalues mimic the behavior of the discrete PSWF with respect to location of zeros. In Example 3 we know that all zeros are on the unit circle due to Theorem 4.2 and 4.3.

In Table II we illustrate the performance of quadrature for different bandlimit c . This table should be compared with [29, Table 1]. The performance of both sets of quadrature is very similar. Yet the quadrature are quite different and can be easily compared using Table III in [29, Table 5]. Although the accuracy is almost identical, approximately 10^{-15} .

TABLE II
Quadrature Performance for Varying Bandlimits

| c | # of node | Max. im. m. e. o. |
|------|-----------|---------------------|
| 20 | 13 | $1.2 \cdot 10^{-7}$ |
| 50 | 24 | $1.1 \cdot 10^{-7}$ |
| 100 | 41 | $1.6 \cdot 10^{-7}$ |
| 200 | 74 | $1.8 \cdot 10^{-7}$ |
| 500 | 171 | $1.4 \cdot 10^{-7}$ |
| 1000 | 331 | $2.4 \cdot 10^{-7}$ |
| 2000 | 651 | $1.2 \cdot 10^{-7}$ |
| 4000 | 1288 | $3.7 \cdot 10^{-7}$ |

5. A NEW ALGORITHM FOR CARATHÉODORY REPRESENTATION

5.1. Algorithm 2

We now describe an algorithm for computing quadrature nodes via a Carathéodory-type approach based on Theorem 4.1. It is easy to see that, although the elements are similar to those with

TABLE III
Quadrature Nodes for Exponentials with Maximum Bandlimit $c = 50$

| Node | Weight |
|-------------------|-------------------|
| -0.99041609489889 | 2.42209284787E-02 |
| -0.95238829377394 | 5.04152570050E-02 |
| -0.89243677566550 | 6.82109308489E-02 |
| -0.81807124037876 | 7.96841731718E-02 |
| -0.73438712699465 | 8.71710040243E-02 |
| -0.64454148960251 | 9.22000859355E-02 |
| -0.55050369342444 | 9.56668891250E-02 |
| -0.45355265507507 | 9.80920675810E-02 |
| -0.35456254990620 | 9.97843340729E-02 |
| -0.25416536256280 | 1.00930070892E-01 |
| -0.15284664158549 | 1.01641529848E-01 |
| -0.05100535080412 | 1.01982696564E-01 |
| 0.05100535080412 | 1.01982696564E-01 |
| 0.15284664158549 | 1.01641529848E-01 |
| 0.25416536256280 | 1.00930070892E-01 |
| 0.35456254990620 | 9.97843340729E-02 |
| 0.45355265507507 | 9.80920675810E-02 |
| 0.55050369342444 | 9.56668891250E-02 |
| 0.64454148960251 | 9.22000859355E-02 |
| 0.73438712699465 | 8.71710040243E-02 |
| 0.81807124037876 | 7.96841731718E-02 |
| 0.89243677566550 | 6.82109308489E-02 |
| 0.95238829377394 | 5.04152570050E-02 |
| 0.99041609489889 | 2.42209284787E-02 |

Pi a enko' method, the co e ponding algo ithm a e . b tantiall diffe ent. We plan to add e implication fo ignal p oce ing in a epa ate pape .

(1) Gi en t_k , the t igonomet ic moment of a mea . e, ν e con t . ct the $(N + 1) \times (N + 1)$ Toeplit mat \mathbf{x} , $T_{N, \nu}$ ith element $(T_N)_{kj} = t_{j-k}$. Thi mat \mathbf{x} , i po iti e de nite and ha a la ge n mbe of mall eigen al e .

(2) Fo a gi en acc. ac , ν e comp te the in e e of the Toeplit mat \mathbf{x} , $T_N - I$. Fo a elf-adjoint Toeplit mat \mathbf{x} , it i . f cient to ol e $(T_N - I)pPon prjV Q p \mathbb{R}^n$

If φ_k e de ne

$$Q(z) = \prod_{k=1}^M (z - \alpha_k) = \sum_{k=0}^M q_k z^k, \tag{5.2}$$

then, fo an pol nomial P of deg ee at mo t $M - 1$,

$$\frac{P(z)}{Q(z)} = \sum_{r=1}^M \frac{P(\alpha_r)}{Q'(\alpha_r)(z - \alpha_r)}.$$

Th , fo $|z| < \min |\alpha_r|^{-1}$,

$$\frac{z^{M-1} P(z^{-1})}{z^M Q(z^{-1})} = \sum_{r=1}^M \frac{P(\alpha_r)}{Q'(\alpha_r)} \sum_{k=0}^{+\infty} \alpha_r^k z^k = \sum_{k=0}^{+\infty} \left(\sum_{r=1}^M \frac{P(\alpha_r)}{Q'(\alpha_r)} \alpha_r^k \right) z^k. \tag{5.3}$$

No φ_k choo e P to be the . niq e pol nomial φ_k ith $P(\alpha_r) = \alpha_r Q$

This algorithm is equivalent to the following factorization of the inverse of the Vandermonde matrix in terms of a diagonal matrix, its transpose V^t , and a triangular Hankel matrix,

$$V^{-1} = \begin{matrix} \frac{1}{Q(z_1)} & \cdots & 0 & & q_1 & q_2 & \cdots & q_M \\ & & \ddots & & q_2 & \cdots & q_M & 0 \\ & & & & \vdots & & & \vdots \\ 0 & \cdots & \frac{1}{Q(z_M)} & & q_M & \cdots & 0 & 0 \end{matrix} V^t \quad (5.5)$$

This description is a particular case of the inversion formulae for L_{ν} -Vandermonde [21] or closely to Vandermonde matrices [9, Corollary 2.1, p. 157]. We can state the result as (see [21, p. 548])

$$V^{-1} = \begin{matrix} & & & -y_2 & -y_3 & \cdots & 1 \\ x_1 & \cdots & 0 & -y_3 & \cdots & 1 & 0 \\ & & \ddots & \vdots & & & \vdots \\ 0 & \cdots & x_M & & & \cdots & 0 \\ & & & 1 & \cdots & 0 & 0 \end{matrix} V^t$$

where the vectors $\mathbf{x} = (x_1, \dots, x_M)^t$ and $\mathbf{y} = (y_1, \dots, y_M)^t$ are a solution of

$$V\mathbf{x} = (0, \dots, 1)^t \quad \text{and} \quad V^t\mathbf{y} = [r^M]_{r=1}^M.$$

Since r are the roots of $Q(z)$, we can take $\mathbf{y} = -(q_0, \dots, q_{M-1})^t$, and if $B(z) = z^M$ in (5.4), then $P(z) = 1$ and $\mathbf{x} = (1/Q(z_1), \dots, 1/Q(z_M))^t$.

Remark 5.1. For Algorithm 5.1, we get obtained the eigenvectors \mathbf{q} corresponding to an eigenvalue λ . Then, step (1) of the Vandermonde algorithm is already accomplished and step (2) can be performed using the FFT. For the moment, the nodes z_k belong to the unit circle and, via the nearest spaced fast Fourier transform, we have a fast algorithm to obtain the eigenvectors.

Remark 5.2. As an example, we use this approach to describe the solution of the Vandermonde system with nodes $z_r = e^{i2\pi(r-1)/M}$, $1 \leq r \leq M$. In this case, $Q(z) = 1 - z^M$

Proof of Theorem 6.1. Let

$$u(y) = \int_{-1}^1 (t)e^{i ty} dt,$$

and, for each m , define the spline of order $2m - 1$ interpolating $u(y)$ at the integers,

$$a(y) = \sum_k u(k)L_{2m-1}(y - k) = \int_{-1}^1 (t)S_{2m-1}(y, e^{it}) dt.$$

By (6.7),

$$|u(y) - a(y)| \leq 3 \int_{-1}^1 (t)|t|^{2m} dt \leq 3^{-2m} \leq \frac{1}{4},$$

where $e^{-1} = \int_{-1}^1 (t) dt$. We choose m such that $3^{-2m} \leq \frac{1}{4}$.

On the other hand, for each N , Theorem 3.7 allows us to represent the moment $u(k)$, $|k| \leq N$,

$$u(k) = \int_{-1}^1 (t)e^{ikt} dt = \sum_{j=1}^N w_j e^{i jk} + w_0 (-1)^k, \tag{6.9}$$

where

$$w_0 = \frac{4^{-1}}{2 + (2 + \frac{1}{3})^N + (2 - \frac{1}{3})^N}. \tag{6.10}$$

Let

$$\tilde{u}(y) = \sum_{j=1}^N w_j e^{i jy};$$

then $u(k) = \tilde{u}(k) + w_0 (-1)^k$ for $|k| \leq N$, and defining

$$\tilde{a}(y) = \sum_k \tilde{u}(k)L_{2m-1}(y - k) = \sum_{j=1}^N w_j S_{2m-1}(y, e^{ij}),$$

(6.7) gives the estimate

$$|\tilde{u}(y) - \tilde{a}(y)| \leq 3 \sum_{j=1}^N w_j |y - j|^{2m} \leq 3^{-2m} (u(0) - w_0) \leq 3^{-2m} \leq \frac{1}{4}.$$

We have hence that $u(y)$ is close to $a(y)$ and $\tilde{u}(y)$ is close to $\tilde{a}(y)$. To finish the proof, we need to show that $|a(y) - \tilde{a}(y)| < \frac{1}{2}$, for $|y| \leq dN + 1$. Now,

$$\begin{aligned} a(y) - \tilde{a}(y) &= \sum_{|k| \leq N} w_0 (-1)^k L_{2m-1}(y - k) + \sum_{|k| > N} (u(k) - \tilde{u}(k)) L_{2m-1}(y - k) \\ &= w_0 S_{2m-1}(y, e^{i \cdot}) + \sum_{|k| > N} (u(k) - \tilde{u}(k) - w_0 (-1)^k) L_{2m-1}(y - k) \end{aligned}$$

and

$$\begin{aligned} |u(k) - \tilde{u}(k) - w_0 (-1)^k| &\leq |u(k)| + |\tilde{u}(k)| + w_0 \leq \sum_{j=0}^N w_j + \sum_{j=1}^N w_j + w_0 \\ 2u(0) &= 2 \leq 1, \end{aligned}$$

where we used (6.9).

Since J_{2n} is an even function, we have

$$v(x) = \int_{-1}^1 \bar{w}(t) J_{2n}(cx) dt. \quad (7.4)$$

Using

$$J_{2n}(t) = \frac{(-1)^n}{t^{2n}}$$

where

$$\tilde{v}_j = \sum_{k=1}^M w_k j(t_k), \tag{7.13}$$

and the nodes t_k and the weights w_k are the same as in (1.4).

For large c , the spectrum of F_c can be divided into two groups. The first group contains approximately $2c/\alpha$ eigenvalues with absolute value close to 1. The second group contains $\log c$ eigenvalues whose absolute values make an exponential transition from 1 to 0. The third group consists of exponentially decaying eigenvalues that are close to zero. For precise statements see [14, 24, 25, 29].

Therefore, it follows from (7.12) that, for the $2c/\alpha$ eigenfunction, the integral in (7.11) is well approximated by the quadrature in (7.13). To prove (7.12), see (7.10), to wit

$$v_j - \tilde{v}_j = \frac{1}{j} \int_{-1}^1 \int_{-1}^1 w(t) e^{ic t d} - \sum_{k=1}^M w_k e^{ic k t} j(t) dt. \tag{7.14}$$

Since $|t| \leq 1$, we have

$$\left| \int_{-1}^1 w(t) e^{ic t d} - \sum_{k=1}^M w_k e^{ic k t} \right|, \tag{7.15}$$

and $j_2 = \text{arXiv:1404.0052 [math]} \text{TD}(0.0052 \text{ Tc}(14 41e012)\text{Tj}67 \text{ 8Tj}/(i)0.9(m \text{ Tm})8.6483 0 \text{ TD-94h5})\text{T1 T}$

In considering bandlimited function $f(x)$ in the PSWF (see [15, 24], and a more recent paper by [16]), the eigenfunctions $\psi_j(x)$ in (7.9) are the eigenfunctions of the

$$j \psi_j =$$

By setting

$$I = w_l \sum_{j=0}^{M-1} j (b/c) j (t_l), \tag{8.18}$$

and observing that $|M|$ and that $|j| \leq |M|$ for $j > M$, we obtain (8.5) and (8.6). ■

We now construct a linear combination of the function $\{e^{ict_l x}\}_{l=1}^M$. First, let us consider the following algebraic eigen value problem,

$$\sum_{l=1}^M w_l e^{ict_l t_l} j (t_l) = j j (t_m), \tag{8.19}$$

where t_l and w_l are the same as in (8.1). By solving (8.19), we find j and $j (t_l)$. We then consider function $j, j = 1, \dots, M$, defined for an x as

$$j (x) = \frac{1}{j} \sum_{l=1}^M w_l e^{icx t_l} j (t_l). \tag{8.20}$$

The function j in (8.20) are linear combination of the exponential functions

matrix, the exit an orthogonal basis of real eigen vectors. Then, computed via (8.22), we assume q_l^j to be a real orthogonal matrix, and then

$$\sum_{j=1}^M \overline{w_l} \cdot j(t_l) \cdot j(t_m) \cdot \overline{w_m} = \delta_{lm} \tag{8.23}$$

and

$$\sum_{l=1}^M j(t_l) w_l \cdot j(t_l) = \delta_{jj} \tag{8.24}$$

We have

$$\int_{-1}^1 j(t) \cdot j(t) dt = \frac{1}{j \cdot j} \sum_{l,l=1}^M w_l w_l \cdot j(t_l) \cdot j(t_l) \int_{-1}^1 e^{ict(t_l+t_l)} dt \tag{8.25}$$

and, from (8.1), we obtain

$$\left| \int_{-1}^1 j(t) \cdot j(t) dt - \frac{1}{j \cdot j} \sum_{l,l=1}^M w_l w_l \cdot j(t_l) \cdot j(t_l) \sum_{k=1}^M w_k e^{ict_k(t_l+t_l)} \right| \leq \frac{2 \sum_{k=1}^M w_k}{|j| |j|} \tag{8.26}$$

Let $\{e^{icx t_l}\}_{l=1}^n$ be a linear combination of the exponential functions $R_k, k = 1, \dots, M$, a

$$R_k(x) = \sum_{l=1}^M r_{kl} e^{icx t_l}, \tag{8.27}$$

we have

$$r_{kl} = \sum_{j=1}^M w_k \int_j(t_k) \frac{1}{j} \int_j(t_l) w_l = \sum_{j=1}^M \overline{w_k} q_k^j \frac{1}{j} q_l^j \overline{w_l}. \tag{8.28}$$

By direct calculation in (8.19) and (8.23), we see that function R_k is an interpolating,

$$R_k(t_m) = \delta_{km}. \tag{8.29}$$

Let us show that the integral of $R_k(t)e^{iat}$, where $|a| < c$, yields a one-point quadrature rule of accuracy $O(\epsilon)$.

PROPOSITION 8.3. For $|a| < c$, let

$$I_k = \int_{-1}^1 R_k(t) e^{iat} dt - w_k e^{iat_k}. \tag{8.30}$$

Then we have

$$|I_k| \leq 2 \frac{\max_{k=1, \dots, M} |w_k|}{\min_{k=1, \dots, M} |w_k|} \epsilon^2, \tag{8.31}$$

where $\epsilon = \sqrt{\sum_{k=1}^M |w_k|^2}$.

Proof. Using (8.27) and (8.29),

$$\sum_{l=1}^M r_{kl} \sum_{m=1}^M w_m e^{ict_m(t_l+a/c)} = \sum_{m=1}^M w_m R_k(t_m) e^{iat_m} = w_k e^{iat_k}, \tag{8.32}$$

and, therefore, I_k in (8.30) can be written as a matrix-vector multiplication $I_k = \sum_{l=1}^M r_{kl} s_l$, where

$$s_l = \int_{-1}^1 e^{ict(t_l+a/c)} dt - \sum_{m=1}^M w_m e^{ict_m(t_l+a/c)}. \tag{8.33}$$

The inequality (8.31) is then obtained via the usual L^2 -norm estimate, taking into account that the matrices q_k^j and q_l^j in (8.28) are orthogonal and that, for function e^{iax} , we have $|a| < c$, (8.1) implies $|s_l| \leq \epsilon^2$. ■

We have observed (via computation) that $\max_{k=1, \dots, M} |w_k| = O(1)$ and $\min_{k=1, \dots, M} |w_k| = O(\epsilon)$ in (8.31), thus yielding in $\epsilon = O(\epsilon)$. Next we deduce a weak estimate showing that the function R_k is close to being an interpolating basis for band-limited exponential.

PROPOSITION 8.4. For every b , $|b| < c$, let us consider the function

$$b(t) = e^{ibt} - \sum_{k=1}^M e^{ibt_k} R_k(t). \tag{8.34}$$

Then, for every $|a| < c$, we have

$$\left| \int_{-1}^1 b(t)e^{iat} dt \right| \leq 1 + M \frac{\max_{k=1, \dots, M} |w_k|}{\min_{k=1, \dots, M} |t_k|}. \tag{8.35}$$

Proof. Using (8.30), we have

$$\int_{-1}^1 b(t)e^{iat} dt = \int_{-1}^1 e^{i(b+a)t} dt - \sum_{k=1}^M w_k e^{i(b+a)t_k} - \sum_{k=1}^M e^{ibt_k} R_k(t), \tag{8.36}$$

hence

$$R_k(t) = \int_{-1}^1 R_k(t)e^{iat} dt - w_k e^{iat_k}. \tag{8.37}$$

Applying (8.1), we obtain

$$\left| \int_{-1}^1 b(t)e^{iat} dt \right| \leq 1 + \overline{M}. \tag{8.38}$$

The estimate (8.35) then follows from Proposition 8.3. ■

Remark 8.2. Using the function R_k , $k = 1, \dots, M$, on a hierarchy of intervals, it is possible to construct a multiresolution basis (for a finite number of scales) similar to multiresolution wavelets. We will consider such construction and its application elsewhere.

8.1. Examples

For the eight

$$R_k(t) = \begin{cases} 1, & t \in [-a, a], \quad a = 1/2, \\ 0, & \text{otherwise,} \end{cases} \tag{8.39}$$

we construct a 30-node quadrature formula so that (8.1) is satisfied with $\epsilon = 10^{-15}$. We

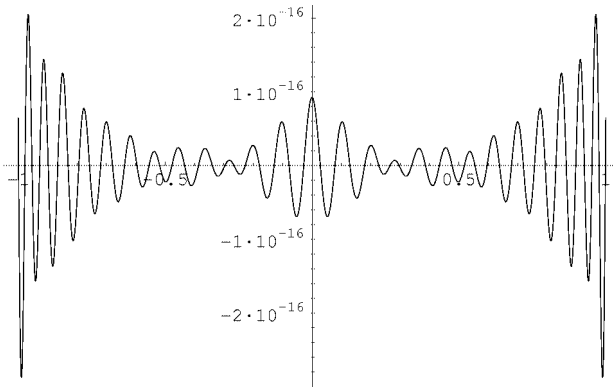


FIG. 9. E_{α} in (8.1) for Example 1.

φ_9 is the Legendre polynomial of degree 9. The α th φ function is not periodic and φ_9 is e^{-x} .

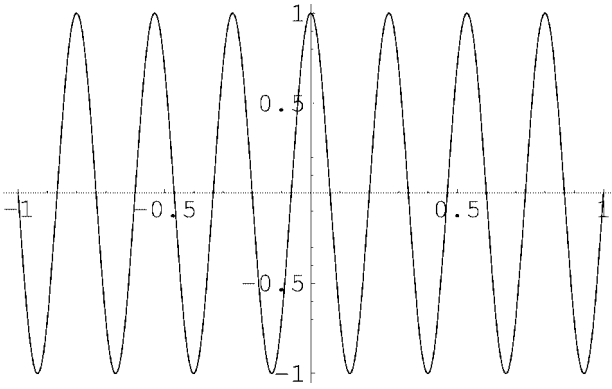


FIG. 11. Function $g_1(t)$ on the interval $[-1, 1]$.

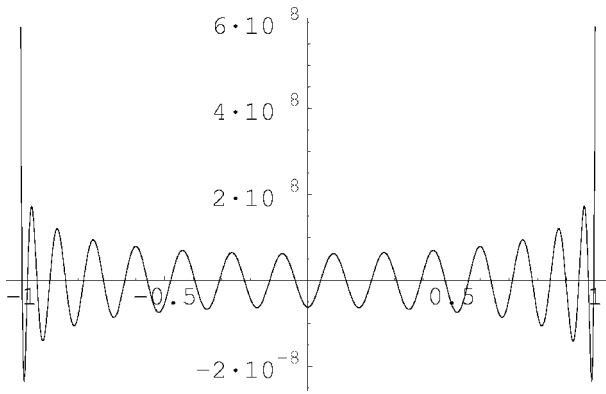


FIG. 12. Difference $g_1(t) - \bar{g}_1(t)$ on the interval $[-1, 1]$.

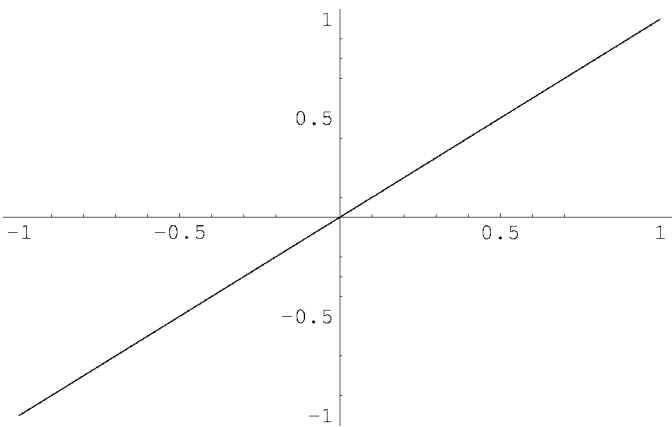


FIG. 13. Function $g_2(t)$ on the interval $[-1, 1]$.

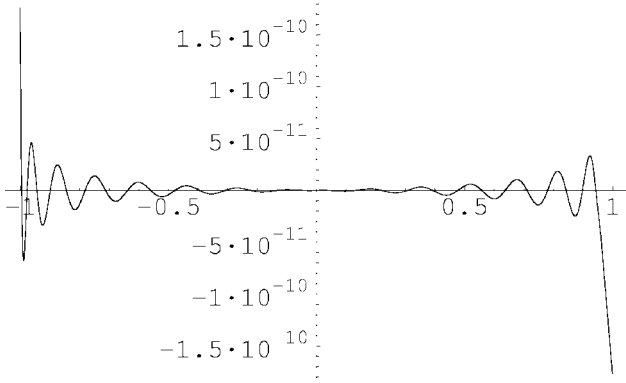


FIG. 14. Difference $g_2(t) - \tilde{g}_2(t)$ on the interval $[-1, 1]$.

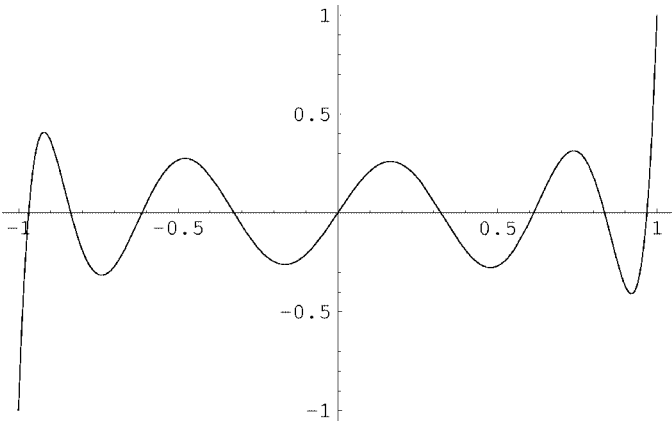


FIG. 15. Function $g_3(t)$ on the interval $[-1, 1]$.

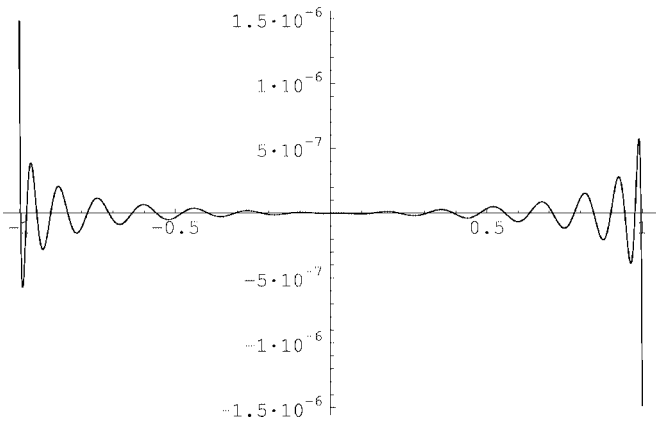


FIG. 16. Difference $g_3(t) - \tilde{g}_3(t)$ on the interval $[-1, 1]$.

exponential decay (see Fig. 1). For small eigen values, the equidistribution is of practical interest.

The remarkable feature of the equidistribution is that the half-nodes outside the support of the measure and, a priori, not, the corresponding eighth are negative and small, roughly of the size of the eigen value. The case corresponding to the smallest eigen value is equivalent to the classical Cauchy distribution.

An application of the new quadrature, especially for approximating and integrating real (essentially) bandlimited functions. We also have connected, using quadrature nodes and for a given precision, an interpolating basis for bandlimited functions on an interval.

In the paper we made a number of observations for which we do not have proof. Let us mention a few. First, it is desirable to have tight uniform estimates for the L -norm of the PSWF (with a fixed bandlimiting constant), ideally, for the eigenfunction associated with more general eighth. Second, we conjecture that in Theorem 4.1, it is not necessary to require distinct roots for the eigenpolynomial since it might be a consequence of the eigen value being simple. We have neither a proof nor a counterexample at this time.

APPENDIX: PROOF OF THEOREM 2.2

We use a technique that goes back to [2] (see [28, Theorem 7.3] and [19, Chapter 5] for more detail) which involves the Fejér kernel,

$$F_L(x) = \sum_{|k| \leq L} \left(1 - \frac{|k|}{L+1}\right) e^{ikx} = \frac{\sin^2\left((L+1)\frac{x}{2}\right)}{(L+1) \sin^2\frac{x}{2}}, \tag{A.1}$$

for real x .

We need the following result.

THEOREM A.1 [19, Theorem 8, Chapter 5]. *For $|k| \leq N$, let*

$$c_k = \sum_{j=1}^M z_j^k,$$

where $z_j \neq 0$ and $|z_j| = 1$. Then, for all L , $0 \leq L \leq N$,

$$(L+1) \int_{-\pi}^{\pi} F_L(x) |c_k|^2 dx = c_0^2 + 2 \sum_{k=1}^L |c_k|^2.$$

Proof. Let $a_k = 1 - |k|/(L+1)$ be the coefficient of the Fejér kernel F_L and write $z_j = e^{i\theta_j}$. Since $z_j \neq 0$ and $F_L(x) \geq 0$ for all x ,

$$\begin{aligned} \sum_{|k| \leq L} a_k |c_k|^2 &= \sum_{|k| \leq L} a_k \sum_{j,l=1}^M z_j^k \bar{z}_l^k \\ &= \sum_{j,l=1}^M z_j^{-l} F_L(z_j^{-l}) = F_L(0) \sum_{j=1}^M 1 = (L+1) \sum_{j=1}^M 1. \end{aligned}$$

The theorem follows because $a_0 = 1$ and $a_k \geq 0$. ■

Proof of Theorem 2.2. We take (2.1) to extend the definition of c_k and $c_{-k} = \overline{c_k}$ for $k = 1, \dots, N$ and $c_0 = \sum_{j=1}^M q_j$. We then define the Toeplitz matrix \mathbf{T}_N , $(\mathbf{T}_N)_{kj} = (c_{j-k})_{0 \leq k, j \leq N}$, and the polynomial

$$Q(z) = \prod_{j=1}^M (z - e^{i\theta_j}) = \sum_{k=0}^M q_k z^k.$$

Then $\mathbf{q} = (q_0, \dots, q_M, 0, \dots, 0)^t$

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