A Multiresolution Method for Numerical Reduction and Homogenization of Nonlinear ODEs

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The multiresolution analysis (MRA) strategy for the reduction of a nonlinear differential equation is a procedure for constructing an equation directly for the coarse scale component of the solution. The MRA homogenization process is a method for building a simpler equation whose solution has the same coarse behavior as the solution to a more complex equation. We present two multiresolution reduction methods for nonlinear differential equations: a numerical procedure and an analytic method. We also discuss one possible appproach to the homogenization method.

I. INTRODUCTION

There are many difficult, interesting, and important problems which incorporate multiple scales and which are prohibitively expensive to solve on the finest scales. In many problems of this kind it is sufficient to find the solution on a coarse scale only. However, we cannot disregard the fine scale contributions as the behavior of the solution on the coarse scale is affected by the fine scales. In these problems it is necessary to obtain a procedure for constructing the equations on a coarse scale that account for the contributions from these scales. This amounts to writing an effective equation for the coarse scale

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component of the solution which can be solved more economically. Alternatively, we might want to construct simpler fine scale equations whose solutions have the same coarse properties as the solutions of more complicated systems. These simpler equations would also be considerably less expensive to solve. These procedures are generally referred to as homogenization, though the specifics of the approaches vary significantly.

An example of a problem which encompasses many scales and which is difficult to solve on the finest scale is molecular dynamics. The highest frequency motion of a polymer chain under the fully coupled set of Newton's equations determines the largest stable integration time step for the system. In the context of long time dynamics the high frequency motions of the system are not of interest but current numerical methods (see [1, 17]) which directly access the low frequency motions of the polymer are *ad hoc* methods, not methods which take into account the effects of the high frequency behavior. The work of Bornemann and Schütte (see [16, 6]) is a notable exception and appears quite promising.

Let us briefly mention several classical approaches to homogenization. The classical theory of homogenization, developed in part by Bensoussan *et al.* [3], Jikov *et al.* [12], Murat [15], and Tartar [18], poses the problem as follows: Given a family of differential operators L_ϵ, indexed by a parameter ϵ, assume that the boundary value problem

$$L_\epsilon u_\epsilon = f \qquad \text{in } \Omega$$

(with u_ϵ subject to the appropriate boundary conditions) is well-posed in a Sobolev space H for all ϵ and that the solutions u_ϵ form a bounded subset of H so that there is a weak limit u_0 in H of the solutions u_ϵ. The small parameter ϵ might represent the relative magnitude of the fine and coarse scales. The problem of homogenization is to find the differential equation that u_0 satisfies and to construct the corresponding differential operator. We call the homogenized operator L_0 and the equation $L_0 u_0 = f$ in Ω the homogenized equation.

There are several methods for solving this problem. A standard technique is to expand the solution in powers of ϵ, to substitute the asymptotic series into the differential equations and associated boundary conditions, and then to recursively solve for the coefficients of the series given the first order approximation to the solution (see [14, 2, 13] for more details). If we consider a probabilistic interpretation of the solutions to elliptic can be used recursively provided that the form of the equation is preserved under the transition. For systems of linear ordinary differential equations a step of the multiresolution reduction procedure consists of changing the coordinate system to split variables into averages and differences (in fact, quite literally in the case of the Haar basis), expressing the differences in terms of the averages, and eliminating the differences from the equations. For systems of linear ODEs there are relatively simple explicit expressions for the coefficients of the resulting reduced system. Because the system is organized so that the form of the equations is preserved, we may apply the reduction step recursively to obtain the reduced system over several scales.

M. Dorobantu [9] and A. Gilbert [10] apply the technique of MRA homogenization to the one-dimensional elliptic problem and derive results which relate the MRA approach to classical homogenization theory. A multiresolution approach to the reduction of elliptic

$$G(t)x(t) + q(t) - b = \int_0^t (F(s)x(s) + p(s))ds, \qquad t \in [0, 1] \tag{2.1}$$

(where b is a complex or real vector) since we can preserve the form of this equation under reduction, while we cannot preserve the form of the corresponding differential equation. To express this integral equation in terms of an operator equation on functions in $L^2([0, 1])$, let **F** and **G** be the operators whose actions on functions aof pointwise multiplication by F and G and let **K** be the integral operator whose kernel K is

$$K(s, t) = \begin{cases} 1, & 0 \leq s \leq t \\ 0, & \text{otherwise.} \end{cases}$$

Then Eq. (2.1) can be rewritten as

$$\mathbf{G}x + q - b = \mathbf{K}(\mathbf{F}x + p).$$

We will use a general MRA of $L^2([0, 1])$ in this discussion. See Appendix B for definitions. We begin with an initial discretization of our integral equation by applying the projection operator P_n and looking for a solution x_n in V_n. This is equivalent to discretizing our problem at a very fine scale. We have

G

$$T_{O,j} \; 5 \; P_j O_{j11} P_j^*, \qquad C_{O,j} \; 5 \; P_j O_{j11} Q_j^*$$

This procedure can be repeated up to n times use the recursion formulas:

$$F_j^{(n)} = T_{F,j} - C_{F,j} R_j^{-1} (B_{G,j} - B_{K,j} T_{F,j} - A_{K,j} B_{F,j}), \tag{2.7}$$

$$G_j^{(n)} = T_{G,j} - C_{K,j} B_{F,j} - (C_{G,j} - C_{K,j} A_{F,j}) R_j^{-1} (B_{G,j} - B_{K,j} T_{F,j} - A_{K,j} B_{F,j}), \tag{2.8}$$

$$q_j^{(n)} = P_j q - C_{K,j} Q_j p - (C_{G,j} - C_{K,j} A_{F,j}) R_j^{-1} (Q_j q - B_{K,j} P_j p - A_{K,j} Q_j p), \tag{2.9}$$

$$p_j^{(n)} = P_j p - C_{F,j} R_j^{-1} (Q_j q - B_{K,j} P_j p - A_{K,j} Q_j p). \tag{2.10}$$

The superscript (n) denotes the resolution level at which we started the reduction procedure and the subscript j denotes the current resolution level.

Let us summarize this discussion in the following proposition.

PROPOSITION II.1. *Suppose we have an equation for* $x_{j+1}^{(n)} = P_{j+1} x_n^{(n)}$ *in* V_{j+1},

$$G_{j+1}^{(n)} x_{j+1}^{(n)} + q_{j+1}^{(n)} - b = K_{j+1} (F_{j+1}^{(n)} x_{j+1}^{(n)} + p_{j+1}^{(n)}),$$

where the operator $R_j = A_{G,j} - B_{K,j} C_{F,j} - A_{K,j} A_{F,j}$ *is invertible. Then we can write an exact effective equation for* $x_j^{(n)} = P_j x_n^{(n)}$ *in* V_j,

$$G_j^{(n)} x_j^{(n)} + q_j^{(n)} - b = K_j (F_j^{(n)} x_j^{(n)} + p_j^{(n)}),$$

using the recursion relations (2.7)–(2.10).

Remark. We initialize the recursion relations with the values

$G_n = P_n \mathbf{G} P_n^*, \quad F_n$

II.2. Nonlinear Reduction Method

solved (if at all). Let us assume that we can solve Eq. (2.13) for d as a function of s and let $\tilde{d}(s)$ denote the solution. We then plug $\tilde{d}(s)$ into Eq. (2.12) to get

$$P\wedge(s, \tilde{d}(s)) = 0$$

which is the reduced equation for the coarse behavior of x. The form of the original system is preserved under this procedure and we may write the recurrence relation for $\wedge_j$ as

$$\wedge_{j-1}(s) = P_{j-1}\wedge_j(s_{j-1}, \tilde{d}_{j-1}(s_{j-1})),$$

where $\tilde{d}_{j-1}(s_{j-1})$ satisfies $Q_{j-1}\wedge_j(s_{j-1}, \tilde{d}_{j-1}(s_{j-1})) = 0$ and $0 \leq j \leq n$.

In this subsection we will give the precise form of the nonlinear system (2.13)–(2.12) in d and s, state conditions for (2.13)–(2.12) under which we can solve for d as a function of s, develop two approaches for solving (2.13)–(2.12) for d (a numerical and an analytic approach), and derive formal recurrence relations for the nonlinear function $\wedge_j$.

We now extend the MRA reduction method to nonlinear ODEs of the form

$$x'(t) = F(t,$$

We will use the MRA of $L^2([0, 1])$ associated with the Haar basis to begin our discretization. We discretize Eq. (2.16) in t by applying the projection operator P_n to Eq. (2.16) and seeking a solution $x_n \in V_n$ to the equation

$$G_n(x_n) = K_n F_n(x_n), \tag{2.17}$$

where

$$G_n(x_n) = P_n \mathbf{G}(x_n), \qquad K_n = P_n \mathbf{K} P_n^*, \qquad \text{and} \qquad F_n(x_n) = P_n \mathbf{F}(x_n).$$

At this point let us work with two consecutive levels and drop the index n indicating the multiresolution level (assume that $d = d_n$). We again modify the Haar basis slightly and normalize the differences by $1/d$. The averages will not be adjusted by any factor. By forming successive averages of Eq. (2.19), we can rewrite Eq. (2.20) in coordinate form as

$$\frac{1}{2}\left(g(x)(2k+1) + g(x)(2k)\right) = \frac{d}{2}\sum_{k'=0}^{2k} f(x)(k') + \frac{d}{4} f(x)(2k+1)$$

$$+ \frac{d}{2}\sum_{k'=0}^{2k-1} f(x)(k') + \frac{d}{4} f(x)(2k). \quad (2.22)$$

In the same manner we rewrite Eq. (2.21) by taking successive differences normalized by the step size d:

$$\frac{1}{d}\left(g(x)(2k+1) - g(x)(2k)\right) = \frac{1}{2}\left(f(x)(2k+1) + f(x)(2k)\right). \quad (2.23)$$

Let us rearrange the right-hand side of Eq. (2.22) as

$$\frac{d}{2}\sum_{k'=0}^{2k} f(x)(k') + \frac{d}{4} f(x)(2k+1) + \frac{d}{2}\sum_{k'=0}^{2k-1} f(x)(k') + d$$

Then we may write the coordinate form of Eqs. (2.20)–(2.21) in a compact form

$$\mathbf{S}g(x)(k) + \frac{\delta^2}{4}\mathbf{D}f(x)(k) = -\delta\sum_{k'=0}^{k-1}\mathbf{S}f(x)(k') + \delta\mathbf{S}f(x)(k) \qquad (2.24))$$

$$\mathbf{D}g(x)(k) = \mathbf{S}f(x)(k). \qquad (2.25)$$

We have split Eq. (2.19) into two sets and now we split the variables accordingly. We define the averages s_{n-1} and the scaled differences d_{n-1} as

$$s_{n-1}(k) = \frac{1}{2}(x_n(2k+1) + x_n(2k)) \qquad \text{and} \qquad d_{n-1}(k) = \frac{1}{\delta}(x_n(2k+1) - x_n(2k)).$$

Notice that since x_n is a piecewise constant function with step width δ_n, then s_{n-1} and d_{n-1} are piecewise constant functions with step width $2\delta_n = \delta_{n-1}$. We will now change variables in Eqs. (2.24) and (2.25) and replace x with

$$x(2k+1) = s(k) + \frac{\delta}{2}d(k) \qquad \text{and} \qquad x(2k) = s(k) - \frac{\delta}{2}d(k).$$

We will abuse our own notation slightly for clarity and denote the change of variables by

$$\mathbf{S}g(s,d)(k) = \frac{1}{2}\left(g\left(s + \frac{\delta}{2}d\right)(2k+1) + g\left(s - \frac{\delta}{2}d\right)(2k)\right)$$

$$\mathbf{D}g(s,d)(k) = \frac{1}{\delta}\left(g\left(s + \frac{\delta}{2}d\right)(2\right.$$

to solve in order to find d in terms of s. Let us assume that we can solve (2.27) for d and let $\tilde{d}$ represent this solution. Notice that Eq. (2.27) is a nonlinear equation for d so that $\tilde{d}$ is a nonlinear function of s. We will discuss how this is implemented numerically in Section III and how this is implemented analytically in Subsection II.3. In the linear case, $\tilde{d}$ is a linear function of s and it can be easily computed explicitly. Provided that we have $\tilde{d}$, we substitute this into Eq. (2.26) and obtain

$$\mathbf{S}g(s,\tilde{d})(k) + \frac{d^2}{4}\mathbf{D}f(s,\tilde{d})(k) = 2d\sum_{k'=0}^{k-1}\mathbf{S}f(s,\tilde{d})(k') + d\mathbf{S}f(s,\tilde{d})(k). \tag{2.28}$$

Observe that we may arrange Eq. (2.28) as

$$g_{n-1}(k)(s_{n-1}) = d_{n-1}\sum_{k'=0}^{k-1} f_{n-1}(k')(s_{n-1}) + \frac{d_{n-1}}{2} f_{n-1}(k)(s_{n-1}), \tag{2.29}$$

where

$$g_{n-1}(k)(s_{n-1}) = \mathbf{S}g_n(k)(s_{n-1},\tilde{d}_{n-1}) + \frac{d_n^2}{4}\mathbf{D}f_n(k)(s_{n-1},\tilde{d}_{n-1}) \tag{2.30}$$

and

$$f_{n-1}(k)(s_{n-1}) = \mathbf{S}f_n(k)(s_{n-1},\tilde{d}_{n-1}). \tag{2.31}$$

In other words, the reduced equation (2.29) is the effective equation for the averages s_{n-1} of x_n. It is important to note that this equation has the same form as the original discretization.

Let us switch now to operator notation to present the recurrence relations for the reduction procedure. We use the solution $\tilde{d}$ of Eq. (2.27) to write Eq. (2.29) in operator form as

$$G^{(n)}_{n-1}(s_{n-1}) = K_{n-1}F^{(n)}_{n-1}(s_{n-1}),$$

where $s_{n-1} = P_{n-1}x$ and the nonlinear operators $G^{(n)}_{n-1}$ and $F^{(n)}_{n-1}$ map V_{n-1} to V_{n-1}. The superscript (n) on the operators denotes the level at which we start the reduction procedure and the subscript $n-1$ denotes the current level of resolution. The operators $G^{(n)}_{n-1}$ and $F^{(n)}_{n-1}$ are defined as the operators which act elementwise according to Eqs. (2.30) and (2.31), respectively. Notice that they have the same form as the operators $G^{(n)}_n$ and $F^{(n)}_n$; both functions $G^{(n)}_{n-1}(s_{n-1})$ and F

where

$$x\sim2k\ 1\ 1!\ 5\ s\sim k!\ 1^{\ d}$$

and

$$f_j{\sim}k!{\sim}s_j! \; 5 \; \mathbf{S}f_{j11}{\sim}k!{\sim}s_j, \; \tilde{d}_j!. \tag{2.36}$$

and similarly for the right side. After expanding both sides of Eq. (2.38) and retaining only terms of order $O(1)$ in d, we have the equation

$$\mathbf{D}g(s)(k) + \mathbf{S}g'(s)(k)d(k) = \mathbf{S}f(s)(k),$$

which we may solve for $\tilde{d}(s)(k)$:

where

$$g_0 \; 5 \; \mathbf{S} g_n$$

$$u_0 \; 5 \; \mathbf{S} f_n$$

$$g_1 \; 5 \; \frac{\tilde{d}}{16} \, \tilde{} \mathbf{D} g^{9}_{n} \; 1 \; \mathbf{S} f^{9}_{n} ! \; 1 \; \frac{1}{16} \mathbf{D} f_n \; 1 \; \frac{\tilde{d}^2}{32} \mathbf{S} g^{0}_{n}$$

$$u_1 \; 5 \; \frac{\tilde{d}}{16} \mathbf{D} f^{9}_{n} \; 1 \; \frac{\tilde{d}^2}{32} \mathbf{S} f^{0}_{n}$$

$$\tilde{d} \; 5 \; \frac{\mathbf{S} f_n \; 2 \; \mathbf{D} g_n}{\mathbf{S} g^{9}_{n}} .$$

In other words, at level j, we arrange the functions $g_j^{(n)}$ and $f_j^{(n)}$ so that they consist of two terms of the appropriate orders and we write recurrence relations for each of these two terms.

Remark. We usually initialize the reduction procedure with the $O(1)$ terms,

$$g_{0,n}^{(n)}(s)(k) = g_n^{(n)}(s)(k), \qquad u_{0,n}^{(n)}(s)(k) = f_n^{(n)}(s)(k),$$

and the $O(d_n^2)$ terms,

$$g_{1,n}^{(n)}(s)(k) = 0, \qquad u_{1,n}^{(n)}(s)(k) = 0.$$

This can be modified, however.

Remark.

$\tilde{d}(k, i)$ form a two-dimensional array. To solve for each $\tilde{d}(k, i)$ we must interpolate among the known values $g(s(k, i))(k)$ since we need to know the value $g(s(k, i)$ 1 $\frac{d}{2}\tilde{d}(k, i))(2k$ 1 $1)$ (and similarly for $g(s(k, i)$ 2 $\frac{d}{2}\tilde{d}(k, i))(2k$

graph that increasing the number of grid points (past 15) will yield no gain in the accuracy of the cubic interpolation method.

III.2.2. Hybrid reduction method. In the second example we will combine the analytic reduction procedure with the numerical procedure. We begin at a very fine resolution $d_{n_0} = 2^{n_0}$ and reduce analytically to a coarser resolution level $d_{n_1} = 2^{n_1}$. From this level we reduce numerically to the final coarse level d_j. The analytic reduction procedure is computationally inexpensive compared to the numerical procedure and we want to take advantage of this efficiency as much as possible. However, we must balance computational expense with accuracy. With this example we will determine the resolution level d_{n_1} at which this balance is achieved. Again we use a separable equation given by

$$x'(t) = x^2(t) \cos(t/e), \qquad x_0$$

The solution to Eq. (3.49) is

$$x\text{~}t!\ 5\ \frac{x_0}{1\ 2\ ex_0\ \sin\text{~}t/e!}.$$

We begin with analytic reduction at resolution $d_{n_0}\ 5\ 2^{2\,10}$. We choose the final resolution level to be $d_j\ 5\ 2^{2\,2}$ and we let n_1, the resolution as which we switch to the numerical

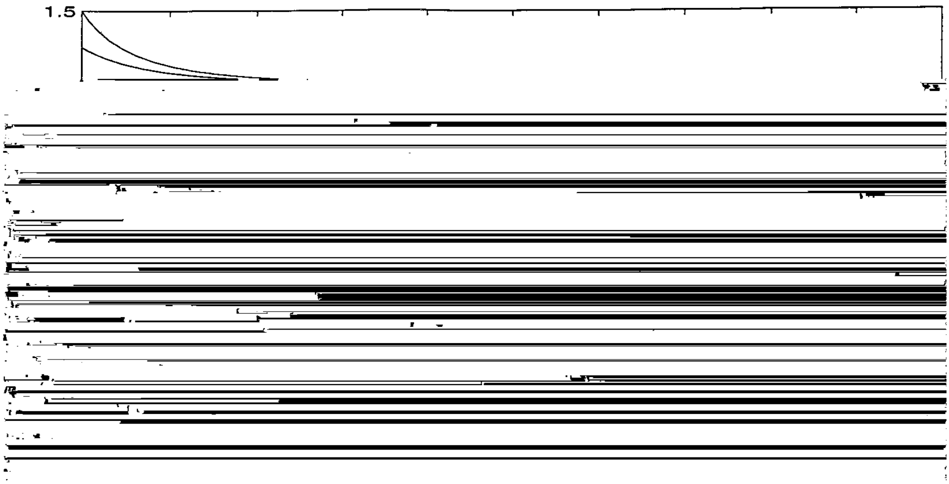

FIG. 3. The flows for Eq. (3.50) with zero forcing.

additional accuracy needed we can use only one relatively more expensive numerical reduction step.

III.2.3. Bifurcation and stability analysis. The third example we will consider is the equation

$$x'(t) = (1 - x^2(t))x(t) + A \sin(t/\varepsilon), \qquad x(0) = x_0, \tag{3.50}$$

where ε is a small parameter associated to the scale of the oscillation in the forcing term. If the amplitude $A = 0$, then the solution $x(t)$ has one unstable equilibrium point at $x_0 = 0$ and two stable equilibria at $x_0 = -1, 1$ (see Fig. 3).

A small perturbation in the forcing term will effect large changes in the asymptotic behavior as t tends to infinity. Therefore, the behavior of the solution on a fine scale will affect the large scale behavior. In particular, if the amplitude A is nonzero but small, then the solution $x(t)$ has three periodic orbits. Two of the periodic orbits are stable while one is unstable (see Fig. 4). As we increase the amplitude A, there is a pitchfork bifurcation—the three periodic orbits merge into one stable periodic orbit (see Fig. 5). We would like to know if we can determine numerically the initial values of these periodic orbits from the reduction procedure and if those periodic solutions are stable or unstable. We will compare these results derived from the reduction procedure with those from the asymptotic expansion of x for initial values near $x_0 = 0$ and for small ε. Let us begin with the asymptotic expansion of x for small values of ε. Assume we have an expansion of the form

$$x(t; \varepsilon) \sim 0 + \varepsilon x_1(t, \tau) + \varepsilon^2 x_2(t, \tau) + \cdots, \tag{3.51}$$

where the fast time scale τ is given by $\tau = t/\varepsilon$. If we substitute the expansion (3.51) into the Eq. (3.50), we have the equation

$$\frac{\partial x_1}{\partial \tau} + \varepsilon\left(\frac{\partial x_1}{\partial t} + \frac{\partial x_2}{\partial \tau}\right) = A \sin \tau + \varepsilon x_1 + O(\varepsilon^2).$$

Equating terms of order one in e, we have $\frac{- x_1}{- t}$ 5 $A \sin$ t, which has the solution $x_1(t,$ t) 5 2A cos t 1 v(t). The function v is determined by a secularity condition which we impose on the terms of order e. Equating the terms of order e gives us the equation

$$\frac{- x_2}{- t} \; 5 \; 2A \cos t \; 1 \; v\text{~}t! \; 2 \; v9\text{~}t!$$

of the expansion (3.51). Therefore, we set this term equal to zero, v 2 v9 5 0, and determine that v

We have derived two approximations for the initial value near x_0 5 0 of the unstable periodic orbit. We will compare these two approximations with the values we determine numerically from the reduction procedure.

We now turn to the numerical reduction procedure. Assume that we can reduce the problem to a resolution level d_j 5 2^{2j} where it no longer depends on time (i.e., or equivalently, we may use the ratio

$$\frac{(\Delta x_e)_{l+1}}{(\Delta x_e)_l} = \frac{g'(x_e) + (\delta/2)\, f'(x_e)}{g'(x_e) - (\delta/2)\, f'(x_e)}.$$

to test the stability of the separation point x_0^*.

Table 4 below lists several values for ε, the amplitude A, and the corresponding average values x_e for the periodic orbits, separation points, and ratios. The separation point which has a corresponding ratio greater than one is the unstable periodic orbit with initial value

where

$$\tilde{G}_1(x)(k) = \frac{1}{24}\left(\frac{\tilde{F}_0(x)(k)}{\tilde{G}'_0(x)(k)}\right)^2 \tilde{G}''_0(x)(k) + \frac{1}{12}\left(\frac{\tilde{F}_0(x)(k)}{\tilde{G}'_0(x)(k)}\right)\tilde{F}''_0(x)(k)$$

and

$$\tilde{F}_1(x)(k) = \frac{1}{24}\left(\frac{\tilde{F}_0(x)(k)}{\tilde{G}'_0(x)(k)}\right)^2 \tilde{F}''_0(x)(k).$$

In other words, on each interval $(k)2^{-j} < t < (k+1)2^{-j}$

approaches; a numerical reduction procedure and a series expansion of the recurrence relations which gives us an analytic reduction procedure.

The numerical procedure requires some *a priori* knowledge of the bounds on the solution since it entails using a range of possible values for the solution and its average behavior and working with all of them together. The accuracy of this scheme increases with the square of the initial resolution but it is computationally feasible for small systems of equations only. We can use the reduced equation, which we compute numerically, to find the periodic orbits of a periodically forced system and to determine the stability of the orbits.

One reduction step in the analytic method consists of expanding the recurrence relations in Taylor series about the averages of the solution. We gather the terms in the series which are all of the same order in d_j, the step size, and identify them as one term in the series so that we have a power series in d_j. Then we write recurrence relations for each term in the series so that the nonlinear functions which determine the solution on the next coarsest scale are themselves power series in the next coarsest step size $d_{j\,2\,1}$. We determine the recurrence relations for an arbitrary term in this power series, show that the recurrence relations converge if applied repeatedly, and investigate the convergence of the power series for linear ODEs.

The homogenization procedure for nonlinear differential equations is a preliminary one.

VI.1. Recursion Relations for Autonomous Equations and the average operator **S** applied to g_n and evaluated at $s_{n\,2\,1}$ yields $g_n(s_{n\,2\,1})($

In other words, if we group the terms in g_j, f_j, and $\tilde{d}_j$ by their order in d_j and if we stipulate that the terms in $g_{j\,2\,1}$ and $f_{j\,2\,1}$ must be grouped in the same fashion, then we can determine the recurrence relations for the coefficients $g_{i,j\,2\,1}(s_{j\,2\,1})(k)$ (i 5 0, . . . , I) in the series expansion of $g_{j\,2\,1}$ (and similarly for the coefficients $u_{i,j\,2\,1}$).

In the program shown in Fig. 6, we first specify the order I of the expansions. In the example program the order is four. Next the four quantities `ge, go, fe,` and `fo` are defined. Notice that we are using the fact that

$$\sim\mathbf{S}g!\sim x!\sim k! \;5\; \frac{1}{2} \sim g\sim x!\sim 2k \;1\; 1! \;1\; g\sim x!\sim 2k!!$$

$$\sim\mathbf{D}g!\sim x!\sim k! \;5\; \frac{1}{d} \sim g\sim x!\sim 2k \;1\; 1! \;2\; g\sim x!\sim 2k!!$$

to express `ge` 5 $g(x)(2k)$, the even-numbered values of $g(x)$, and `go` 5 $g(x)(2k$ 1 1), the odd-numbered values. The step-size d is accorded the variable `h` in the program. Next we form the two sides of the equation `QG 2 QF 5 0` which determines $\hat{d}$; at the same time we substitute `x(2k 1 1) 5 s(k) 1 h/2d(k)` and `x(2k) 5 s(k) 2 h/2d(k)` into `ge` and `fe` (respectively, `go` and `fo`). Into the expression `QG 2 QF`, we substitute the series expansion for $\tilde{d}$,

```
d5sum(d(i)*(2*h)(2*i),   i50 . . . ord).
```

We expand the expression `QG 2 QF` in a Taylor series and we peel off the zeroth-order coefficient in `h` and solve for `d(0),` which gives us the first term in our expansion for $\tilde{d}$. This is the recurrence relation for h

Recall that the recurrence relation for f_j is f_j 5 **S**$f_{j\,1\,1}$ and notice that **S**$f_{j\,1\,1}$ is the same as `QF` so we simply substitute the expansion for $\tilde{d}$ into `QF.` Then we let `h 5 h/2` to adjust the resolution size for the next step and finally expand the expression in a Taylor series. (Recall that $g_{j\,1\,1}$ and $f_{j\,1\,1}$ are expanded in powers of $d_{j\,1\,1}$ 5 $d_j/2$ and g_j and f_j are expanded in powers of d_j.) To determine the recurrence relation for the coefficient $u_i(s)(k)$, we peel off the ith coefficient (for i # `ord`):

```
coeff(newf, h, i);.
```

The recurrence relation for g_j is given by g_j 5 **S**$g_{j\,1\,1}$ 1 $h^2/4$**D**$f_{j\,1\,1}$ which we denote by `PG.` Again we substitute `x(2k 1 1) 5 s(k) 1 h/2d(k)` and `x(2k) 5 s(k) 2 h/2d(k)` into `ge` and `fe` (respectively, `go` and `fo`) and we substitute the expansion for $\tilde{d}$ into `PG`. Finally we rescale `h` and expand `PG` in a Taylor series. We determine recurrence

1} is an orthonormal basis for each subspace V_j. Here $\phi_{j,k}$ denotes a translation and dilation of ϕ:

$$\phi_{j,k} = 2^{j/2}\phi(2^j x - k).$$

As a consequence of the above properties, there is an orthonormal wavelet basis

$$\{\psi_{j,k} \mid j \geq 0,\ k = 0, \ldots, 2^j - 1\}$$

of $L^2([0, 1])$, $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$, such that for all f in $L^2([0, 1])$

$$P_{j+1}f = P_j f + \sum_{k=0}^{2^j-1} \langle f, \psi_{j,k}\rangle \psi_{j,k}.$$

If we define W_j to be the orthogonal complement of V_j in V_{j+1}, then

$$V_{j+1} = V_j \oplus W_j.$$

We have, for each fixed j, an orthonormal basis $\{\psi_{j,k} \mid k = 0, \ldots, 2^j - 1\}$ for W_j. Finally, we may decompose $L^2([0, 1])$ into a direct sum

$$L^2([0, 1]) = V_0 \oplus \bigoplus_{j \geq 0} W_j.$$

The operator Q_j