i. [8 pts] Are there any points in the plate that are locally thicker or thinner than their nearby surroundings? If so, where are they? If not, explain why not.

ii. [5 pts] Is there a thinnest part of the plate? Do not nd it, simply answer YES or NO and give a brief explanation justifying your answer.

SOLUTION:

(a) We need the gradient of the magnetic eld.

$$r\ B = \left(\frac{1}{xyz}(yz); \frac{1}{xyz}(xz); \frac{1}{xyz}(xy) \right) = \left(\frac{1}{x}; \frac{1}{y}; \frac{1}{z} \right)$$

The maximum rate of change of the magnetic eld occurs in the direction of the gradient so the ship should be aimed in the direction

$$r\ B(1;1;2) = \left(1;1;\frac{1}{2} \right) = i + j + \frac{1}{2}k$$

and the maximum rate of change of the magnetic eld will be given by

$$kr\ B(1;1;2)k = \sqrt{1^2 + 1^2 + \left(\frac{1}{2}\right)^2} = \frac{3}{2}$$

(b) i. We need to nd and classify the critical points.

$$h_x = \ 2xe^y = 0 \ =) \ x = 0$$

$$h_y = e^y\ \ 2y + y^2 \ \ x^2 \ = 0 \ =) \ 2y + y^2 = 0 \ \ (\text{since } x = 0) \ =) \ y = 0;\ \ 2$$

Critical points are $(0;0); (0;\ \ 2)$. Now apply the Second Derivatives Test.

$$D(0;0) = h_{xx}(0;0)h_{yy}(0;0) \ \ [h_{xy}(0;0)]^2 = (\ \ 2)(2) \ \ 0^2 = \ \ 4 < 0 \ =) \ (0;0) \text{ is a saddle point}$$

$$D(0;\ \ 2) = h_{xx}(0;\ \ 2)h_{yy}(0;\ \ 2) \ \ [h_{xy}(0;\ \ 2)]^2 = \ \ 2e^{\ \ 2} \ \ 2e^{\ \ 2} \ \ 0^2 = 4e^{\ \ 4} > 0$$

$$\text{and } h_{xx}(0;\ \ 2) = \ \ 2e^{\ \ 2} < 0 \ =) \ h(0;\ \ 2) \text{ is a local maximum}$$

The thickness is a local maximum at $(0;\ \ 2)$ so there is a point that is locally thicker than its nearby surroundings. There are no points in the plate that are locally thinner than their surroundings.

ii. YES. The thickness is a continuous function and the plate is a closed, bounded region so the Extreme Value Theorem applies. Since the interior critical points are a saddle and a local maximum, the thinnest part of the plate will be on the boundary.

3. [2350/031523 (25 pts)] Let $g(x;y) = \cos(\ \ xy\ \ ) +$

(b)

$$dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy = \left[-\pi y \sin(\pi xy) + y^2\right] dx + [-\pi x \sin(\pi xy) + 2xy]\, dy$$

At $(1;1)$ we have $dg = 1\,dx + 2\,dy$ so that g is more sensitive to small changes in y.

(c) You arrive at the point $\left(\frac{1}{2};1\right)$ when $t = 4$ and the rate of change of temperature with respect to time is given by

$$\frac{dg}{dt} = \nabla g \cdot r'(t) = \left\langle -\pi y \sin(\pi xy) + y^2; -\pi x \sin(\pi xy) + 2xy \right\rangle \cdot \left\langle -\frac{1}{2}t^{-3/2}; \frac{t}{8}\right\rangle$$

$$\Rightarrow \left.\frac{dg}{dt}\right|_{t=4} = \left\langle -\pi(1)\sin(\pi/2) + 1^2; -\pi(1/2)\sin(\pi/2) + 2(1/2)(1)\right\rangle \cdot \left\langle -\frac{1}{2}4^{-3/2}; \frac{4}{8}\right\rangle$$

$$= \left\langle 1-\pi; 1-\frac{\pi}{2}\right\rangle \cdot \left\langle -\frac{1}{16}; \frac{1}{2}\right\rangle = -\frac{1}{16} + \frac{\pi}{16} + \frac{8}{16} - \frac{4\pi}{16} = \frac{1}{16}(7-3\pi)$$

Since this is negative, the temperature is decreasing at a rate of $\frac{1}{16}(7-3\pi)$

4. [2350/031523 (16 pts)] A skateboard park has a new track in the form of the hyperbola $\frac{1}{4}x^2 - y^2 = 12$. You are standing at the point $(x;y) = (0;10)$

Solution:

(a)

$$f \quad 1; \quad \tfrac{1}{2} \quad = 1$$

$$f_x(x;y) = \quad 2xe^{\ (x^2+2y)} \quad =) \quad f_x \quad 1; \quad \tfrac{1}{2} \quad = \quad 2$$

$$f_{xx}(x;y) = \quad 2e^{\ (x^2+2y)} \quad 2x^2 + 1 \quad =) \quad f_{xx} \quad 1; \quad \tfrac{1}{2} \quad = 2$$

$$f_{xy}(x;y) = 4\,xe^{\ (x^2+2y)} \quad =) \quad f_{xy} \quad 1; \quad {}^{1}$$