Work out the following problems and simplify your answers.

1. (30 pts) Determine if the following series converge or diverge. Fully justify your answer and state which test you used.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ (b) $\displaystyle\sum_{n=1}^{\infty} \sqrt[3]{\frac{n+5}{n^7+n^2}}$ (c) $\displaystyle\sum_{n=0}^{\infty} \frac{3^{n+1} n!}{(2n)!}$

Solution:

(a) First, we note that we can use the integral test since $\dfrac{1}{n(\ln n)^2}$ is positive and decreasing for $n \geq 2$. Using the integral test, we have

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{t\to\infty} \int_2^{t} \frac{1}{x(\ln x)^2}\,dx = \lim_{t\to\infty} \int_{\ln 2}^{t} \frac{1}{u^2}\,du = -\lim_{t\to\infty} \frac{1}{u}\Big|_{\ln 2}^{t} = \lim_{t\to\infty} \frac{1}{\ln 2} - \frac{1}{t} = \frac{1}{\ln 2}$$

showing that the integral converges. By the integral test, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges.

(b) We can show convergence using the direct comparison test. By making the denominator smaller, we get the following useful inequality

$$0 < \sqrt[3]{\frac{n+5}{n^7+n^2}} < \sqrt[3]{\frac{n+5}{n^7}} = \sqrt[3]{\frac{n}{n^7}} + \sqrt[3]{\frac{5}{n^7}} = \frac{1}{n^{4/3}} + \frac{5}{n^{7/3}}.$$

Next, we note that

$$\sum_{n=1}^{\infty} \frac{1}{n^{4/3}} + \frac{5}{n^{7/3}} = \sum_{n=1}^{\infty} \frac{1}{n^{4/3}} + \sum_{n=1}^{\infty} \frac{5}{n^{7/3}}$$

is the sum of two convergent p-series with $p = 4/3 > 1$ and $p = 7/3 > 1$ respectively and is thus convergent itself. To finish up, by the direct comparison test $\displaystyle\sum_{n=1}^{\infty} \sqrt[3]{\frac{n+5}{n^7+n^2}}$ converges.

(c) Using the ratio test, we have

$$\left|\frac{a_{n+1}}{a_n}\right|$$

2. (15 pts) Find the sum of the following series $\sum_{n=0}^{\infty} \frac{3}{n^2+3n+2}$. (Hint: use partial fractions)

Solution: We start with partial fractions as

$$\frac{3}{n^2+3n+2} = \frac{3}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2}.$$

Solving for A and B yields $A = 3$ and $B = -3$. Assuming that we have a telescoping series, we compute the partial sums as

$$s_n = \sum_{i=0}^{n} \frac{3}{n^2+3n+2} = \sum_{i=0}^{n} \frac{3}{n+1} - \frac{3}{n+2}$$

$= \frac{3}{1} - \frac{3}{2} + \frac{3}{2}$ 3+ $\frac{3}{3}$ 82=m $\frac{3}{3}$ 9.548 $\frac{3}{4}$ -14.048 Td $\frac{3}{n+1}$ [(+)]TJ/F400 $\frac{3}{n+2}$ 0]TJ 11.712303 0 Td [(/F391J 11.71

$= \frac{3}{1} - 3$

4.

5. (20 pts) Compute the following stating the radius of convergence for each part.

   (a) Write out the power series centered at 0 for $\dfrac{1}{1-x}$.

   (b) Find a power series centered at 0 of $\dfrac{1}{3+x}$.

   (c) By integrating, find a power series for $\ln(3+x)$.

   (d) Find the power series for $x^2 \ln(3+x)$.

Solution:

(a) $\dfrac{1}{1-x}$ is our simple geometric series given by

$$\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad R = 1.}$$

(b) Using the geometric series, we have

$$\frac{1}{3+x} = \frac{1}{3}\frac{1}{1+\frac{x}{3}} = \frac{1}{3}\frac{1}{1-\left(-\frac{x}{3}\right)} = \frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{x}{3}\right)^n = \boxed{\sum_{n=0}^{\infty}(-1)^n\frac{1}{3^{n+1}}x^n.}$$

The radius of convergence can be found by noting that we only have convergence when

$$\left|-\frac{x}{3}\right| < 1 \implies |x| < 3 \implies \boxed{R = 3.}$$

x).