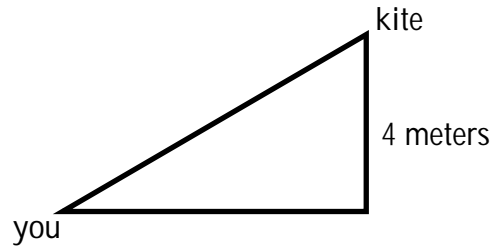


1. (30 points) The following problems are not related.

(a)

- i. the maximum error for the area of the square;
 - ii. the relative error for the area of the square.
- (b) (12 points) You are flying a kite which has a constant height of 4 meters above the ground. The wind is carrying the kite horizontally away from you, and you have to let out string at a rate of 2 meters/minute. What is the horizontal speed of the kite when you have let out 5 meters of string?



Solution:

- (a) i. The area of a square is given by $A(h) = h^2$, so we have that

$$dA = 2hdh = (2)(3)(0.1) = 0.6:$$

Hence, the maximum error for the area of the square is 0.6 cm^2 in this situation.

- ii. For a side length measurement of 3 cm, the area is 9 cm^2 , so the relative error for the area is

$$\frac{dA}{A} = \frac{0.6}{9} = \frac{3}{5} \cdot \frac{1}{9} = \frac{1}{15} \quad 6.\overline{66}\%:$$

- (b) Letting z be the hypotenuse (the distance from you to the kite), and x the horizontal distance, we know that

$$z^2 = x^2 + 4^2 \Rightarrow 2z \frac{dz}{dt} = 2x \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{z}{x} \frac{dz}{dt}:$$

In order to get a value for $\frac{dx}{dt}$, we first need to get the value of x when $z = 5$:

$$5^2 = x^2 + 4^2 \Rightarrow 25 = x^2 + 16 \Rightarrow 9 = x^2 \Rightarrow x = 3:$$

Hence, when $z = 5$, we have that

$$\frac{dx}{dt} = \frac{z}{x} \frac{dz}{dt} = \frac{5}{3} (2) = \frac{10}{3} \text{ meters}=\text{min}:$$

3. (16 points) Consider the function $s(x) = x^3 + 3x + 2$.

- (a) Find the critical numbers of $s(x)$.
- (b) Use the first derivative test to determine the points where $s(x)$ has a local maximum or local minimum. *Give your answer as ordered pairs $(x; y)$.*
- (c) Find the absolute maximum and minimum values for the function $s(x)$ on the interval $[0; 2]$.

Solution:

(a) To find the critical numbers, first take the derivative

$$s'(x) = 3x^2 + 3:$$

Since the domain of $s'(x)$ is $(-1; 1)$, the only critical numbers are solutions to the equation

$$0 = 3x^2 + 3;$$

and hence

$$0 = 3x^2 + 3 \Rightarrow 3x^2 = 3 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1:$$

So $x = \pm 1$ are the only critical numbers. The function values at the critical numbers are $s(-1) = 0$, $s(1) = 4$

(b) In order to determine whether each critical number $x = \pm 1$ is a local maximum, minimum, or neither, we apply the first derivative test to the intervals $(-1; 1)$, $($

which implies that $x = 2$. Hence, we have to find the tangent line at the point $(2; 0)$. Plugging these values into the formula for $\frac{dy}{dx}$, we find that

$$\frac{dy}{dx} \Big|_{(x,y)=(2,0)} = \frac{1-2 \cdot 0}{0+1} = \frac{1}{2}.$$

Then an equation for the tangent line to the curve at $(2; 0)$ is given by

$$y = \frac{1}{2}(x - 2):$$

5. (16 points) Consider the function $f(x) = \frac{1}{x}$ on the interval $[2; 4]$.

- (a) (8 points) State the Mean Value Theorem and verify that $f(x)$ satisfies the hypotheses on the given interval.
- (b) (8 points) Find all numbers c that satisfy the conclusion of the Mean Value Theorem for $f(x)$ on the interval $[2; 4]$.

Solution:

- (a) If $f(x)$ is continuous on $[a; b]$ and differentiable on $(a; b)$, then there is a c in the interval $(a; b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The only value where $f(x)$ is discontinuous is $x = 0$, so $f(x)$ is continuous on $[2; 4]$. The function $f(x)$ is differentiable on $[2; 4]$, since $f'(x) = -\frac{1}{x^2}$, which is undefined only at $x = 0$.