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1. (42 pts) The following problems are unrelated.

   (a) Find the derivative of $y = \sqrt{5x^2 - \sin x}$.

   (b) Evaluate $\displaystyle\int \frac{\arcsin(x)}{\sqrt{1 - x^2}}\, dx$.

   (c) Evaluate $\displaystyle\int_0^{\ln\sqrt{3}} \frac{e^x}{1 + e^{2x}}\, dx$.

   (d) Estimate the value of $\displaystyle\int_1^5 \ln\left(\frac{x}{x+1}\right) dx$ using a Riemann sum with right endpoints and $n = 4$ rectangles of equal width. Express your answer in terms of a single logarithm.

   (e) Evaluate $\displaystyle\lim_{x \to \infty} 2x \sinh\left(\frac{3}{x}\right)$.

Solution:

(a) $\displaystyle\frac{dy}{dx} = \frac{10x - \cos x}{2\sqrt{5x^2 - \sin x}}.$

(b) We will use the substitution $u = \arcsin(x)$. So, $du = \frac{1}{\sqrt{1-x^2}} dx.$

$$\begin{aligned}\int \frac{\arcsin(x)}{\sqrt{1 - x^2}}\, dx &= \int u\, du \\ &= \frac{1}{2}u^2 + C \\ &= \frac{1}{2}(\arcsin(x))^2 + C.\end{aligned}$$

(c) We will use the substitution $u = e^x$. So, $du = e^x\, dx$, the new upper limit of integration will be $u = e^{\ln\sqrt{3}}$

$$
\begin{aligned}
\lim_{x \to \infty} 2x \sinh\left(\frac{3}{x}\right) &= \lim_{x \to \infty} \frac{2\sinh\left(\frac{3}{x}\right)}{\frac{1}{x}} \\
&=^{H} \lim_{x \to \infty} \frac{\cosh\left(\frac{3}{x}\right)\left(\frac{-6}{x^2}\right)}{\frac{-1}{x^2}} \\
&= \lim_{x \to \infty} 6\cosh\left(\frac{3}{x}\right) \\
&= 6.
\end{aligned}
$$

2. (12 pts)

(a) State the definition of continuity of a function, $f(x)$, at a point, $x = a.$

(b) Now consider the function $f(x)$ defined on $[-1, 1]$ by

$$
f(x) = \begin{cases} 2\sin^{-1}(x) & \text{if } x < \frac{1}{2} \\ c & \text{if } x = \frac{1}{2} \\ \cos^{-1}(x) & \text{if } x > \frac{1}{2}. \end{cases}
$$

Is there a value of c that makes f continuous at $x = \frac{1}{2}$

(a) We see that $h'(x) = \frac{1}{\sqrt{x}} - \frac{1}{4}$. Since $h'(x)$ exists on the interior of the domain of $h(x)$ and $h'(x) = 0$ has only a solution of $x = 16$, then $x = 16$ is the only critical number of $h(x)$. Since $h'(x) > 0$ when $0 < x < 16$ and $h'(x) < 0$ when $x > 16$, then $h(x)$ has a local maximum value at $x = 16$, and no local minimum values.

(b) $r'(x) = \left(2\sqrt{\tan(x)} - \frac{1}{4}\tan(x)\right)\sec^2(x).$

4. (32 pts) Consider $s(x) = \frac{e^{2x}}{3 - e^{2x}}$.

(a) Determine $s'(1)$. (Your final answer should be in terms of e.)

(b) Determine the inverse of $s(x)$. Be sure to label your final answer as $s^{-1}(x)$. (You may assume without proof that $s(x)$ is one-to-one.)

(c) Determine all horizontal asymptotes of $s(x)$. Justify each with the appropriate limit.

(d) Determine all vertical asymptotes of $s(x)$. Justify each with the appropriate limit.

Solution:

Note: For many of these problems, you may alternatively note that

$$s(x) = \frac{e^{2x}}{3 - e^{2x}} \cdot \frac{e^{-2x}}{e^{-2x}} = \frac{1}{3e^{-2x} - 1}$$

before proceeding. This will lead to solutions equivalent to the below.

(a)

$$\begin{aligned} s'(x) &= \frac{(3 - e^{2x})2e^{2x} - e^{2x}(-2e^{2x})}{(3 - e^{2x})^2} \\ &= \frac{6e^{2x}}{(3 - e^{2x})^2} \\ s'(1) &= \frac{6e^2}{(3 - e^2)^2}. \end{aligned}$$

(b)

$$\begin{aligned} y &= \frac{e^{2x}}{3 - e^{2x}} \\ x &= \frac{e^{2y}}{3 - e^{2y}} \\ (3 - e^{2y})x &= e^{2y} \\ e^{2y}(-x - 1) &= -3x \\ e^{2y} &= \frac{3x}{x + 1} \\ y &= \frac{1}{2}\ln\left(\frac{3x}{x + 1}\right) \\ s^{-1}(x) &= \frac{1}{2}\ln\left(\frac{3x}{x + 1}\right). \end{aligned}$$

(c) The following limits is an $\frac{1}{1}$

Solving for the desired rate and plugging in the known values at that moment, we have

$$\begin{aligned} \frac{d}{dt} &= \frac{\cos^2(\ )}{9}\frac{dx}{dt} \\ &= \frac{(9=15)^2}{9} \cdot 3 \\ &= \frac{3}{25} \text{ radians per second}: \end{aligned}$$

Note: We used the Pythagorean theorem to find the length of the hypotenuse and determine cos at that moment.