1. (28 points) The following problems are not related.

   (a) Find the general antiderivative of $g(x) = \dfrac{e^{\sqrt{x}}}{\sqrt{x}}$.

   (b) Use logarithmic differentiation to find the derivative of $y = (x^4 + 1)^x$. You do not need to simplify your answer.

   (c) Find the derivative of $f(x) = \displaystyle\int_0^{\cos(x)} \sqrt{1 + t^3}\, dt$.

   Solution:

   (a) Setting $u = \sqrt{x}$ implies that $2\,du = \dfrac{dx}{\sqrt{x}}$, so

   $$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx = 2 \int e^u\, du = 2e^u + C = 2e^{\sqrt{x}} + C.$$

   (b) Taking logarithms yields

   $$\ln(y) = x \ln(x^4 + 1);$$

   and differentiating with respect to x gives

   $$\frac{1}{y}\frac{dy}{dx} = \ln(x^4 + 1) + \frac{4x^4}{x^4 + 1}.$$

   Solving for dy/dx in terms of x gives

   $$\frac{dy}{dx} = (x^4 + 1)^x \left[\ln(x^4 + 1) + \frac{4x^4}{x^4 + 1} \right].$$

   (c) $f'(x) = -\sin(x)\sqrt{1 + \cos^3(x)}$.

2. (26 points) The following problems are not related:

   (a) Find the derivative of $f(x) = \ln\left(\tan^{-1}(x)\right)$.

   (b) Evaluate the definite integral $\displaystyle\int_0^{\ln(3)} \sinh(x)\cosh(x)\, dx$, and fully simplify your answer.

   (c) Determine the value of the limit $\displaystyle\lim_{x \to 0^+} x^2 \ln(x^2)$.

   Solution:

   (a) $f'(x) = \dfrac{1}{1 + x^2} \dfrac{1}{\tan^{-1}(x)}$

(b) Making $u = \sinh(x)$ implies that $du = \cosh(x)\,dx$, and the bounds become

$$u(0) = \sinh(0) = 0$$
$$u(\ln(3)) = \sinh(\ln(3)) = \frac{1}{2}\left(e^{\ln(3)} - e^{-\ln(3)}\right) = \frac{1}{2}\left(3 - \frac{1}{3}\right) = \frac{4}{3}.$$

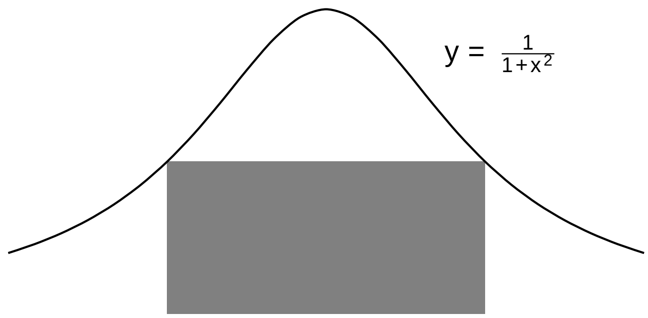
Evaluating the integral:

$$\begin{aligned}\int_0^{\ln(3)} \sinh(x)\cosh(x)\,dx &= \int_0^{4/3} u\,du \\ &= \frac{1}{2}u^2\Big|_0^{4/3} \\ &= \frac{1}{2}\left(\frac{4}{3}\right)^2 \\ &= \frac{8}{9}.\end{aligned}$$

(c) The limit yields the indeterminate form $0 \cdot (-\infty)$, so we apply L'Hôpital's rule:

$$\begin{aligned}\lim_{x\to 0^+} x^2\ln(x^2) &= \lim_{x\to 0^+} \frac{\ln(x^2)}{1/x^2} \\ &\overset{LH}{=} \lim_{x\to 0^+} \frac{2x/x^2}{-2/x^3} \\ &= \lim_{x\to 0^+} \left(\frac{2}{x}\right)\left(-\frac{x^3}{2}\right) \\ &= -\lim_{x\to 0^+} x^2 \\ &= 0.\end{aligned}$$

3. (16 points) Find the area of the largest rectangle which is symmetric around the y-axis, bounded below by the x-axis, and which has two corners touching the graph of $f(x) = \dfrac{1}{1+x^2}$.



$y = \frac{1}{1+x^2}$

Solution:

(a) Since $a(t) = -1$ and $v(0) = 2$, integrating $a(t)$ yields $v(t) = -t + 2$.

(b) Integrating $v(t)$ and using the fact that $s(0) = 0$ yields $s(t) = -\frac{1}{2}t^2 + 2t$. Hence, the total displacement is $s(5) = -\frac{5}{2}$ feet.

(c) Note that the bug changes direction when

(a) Find the domain of the function, and give your answer in interval notation.

(b) Find all horizontal asymptotes of $g(x)$, and justify your answer with limits.

Solution:

(a) The domain of $\arctan(x)$ is $(-\infty, \infty)$, and the domain of $\frac{1}{x^2-4}$ is $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$. Hence, the domain of $g(x)$ is also given by

$$(-\infty, -2) \cup (-2, 2) \cup (2, \infty).$$

(b)

$$\begin{aligned}\lim_{x\to\pm\infty} g(x) &= \lim_{x\to\pm\infty} \left(\arctan(x) + \frac{1}{x^2-4}\right) \\ &= \lim_{x\to\pm\infty} \arctan(x) + \underbrace{\lim_{x\to\pm\infty} \frac{1}{x^2-4}}_{=0} \\ &= \pm\frac{\pi}{2}.\end{aligned}$$

Therefore, $g(x)$ has horizontal asymptotes at $\frac{\pi}{2}$ (when $x \to \infty$) and $-\frac{\pi}{2}$ (when $x \to -\infty$).

7. (16 points) The half-life of the chemical element cobalt-56 is approximately 77 days. Suppose we have a 10 milligram sample of cobalt-56..

(a) Find a formula for the mass of cobalt-56 remaining after t days.

(b) How long will it take for only 1 milligram of cobalt-56 to remain in the sample? It is OK for your answer to have a logarithm in it.

Solution:

(a) Suppose that $m(t)$ is the mass of cobalt-56 remaining after t days. Using the law of natural decay, we know that

$$m(t) = 10e^{kt},$$

so we need to solve for the constant k. Using the information about the half-life we have

$$5 = 10e^{k(77)} \implies \frac{1}{2} = e^{77k} \implies \ln(1/2) = 77k \implies k = \frac{\ln(1/2)}{77}.$$

Hence, the following formulas for $m(t)$ are all valid:

$$\begin{aligned}m(t) &= 10e^{(\ln(1/2)/77)t} \\ &= 10\left(\frac{1}{2}\right)^{t/77} \\ &= 10e^{-(\ln(2)/77)t} \\ &= 10(2)^{-t/77}.\end{aligned}$$

(b) A single milligram of cobalt-56 will remain when

$$1 = 10\left(\frac{1}{2}\right)^{t=77} \implies \frac{\ln(1=10)}{\ln(1=2)} = \frac{t}{77} \implies t = \frac{-77\ln(10)}{-\ln(2)} \implies t = \frac{77\ln(10)}{\ln(2)}$$

This time is approximately 255:79 days.

8. (16 points) For each of the following questions, give a short justification for your answer.

(a) If $f(x)$ is an odd function and $\int_{-3}^{0} f(x)\,dx = \quad + 1$, find $\int_{-3}^{3} f(x)\,dx$.

(b) Find the absolute minimum of the function $f(x) = x \cdot 2^x$, if it exists.

(c) Evaluate the /F37 10.9091 Tf 6.515 Tf 0.26[(days)](=dx)]TJ/F61 10.9091 To)