# APPM 1345
# Exam 3
# Spring 2024

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| **Name** | |
| **Instructor** Richard McNamara | **Section** 150 |

1. (23 pts) Parts (a) and (b) are unrelated.

(a) Find the inverse function of $f(x) = \dfrac{\ln(2x)}{1 + \ln(2x)}$ for $x \quad \dfrac{1}{2}$.

Express your answer in the form $f^{-1}(x)$. (You do not have to identify the inverse function's domain.)

**Solution:**

$$y = \frac{\ln(2x)}{1 + \ln(2x)}$$

$$y[1 + \ln(2x)] = \ln(2x)$$

$$y + y\ln(2x) = \ln(2x)$$

$$(y - 1)\ln(2x) = -y$$

$$\ln(2x) = \frac{y}{1 - y}$$

$$2x = e^{y/(1-y)}$$

$$x = \frac{1}{2}e^{y/(1-y)}$$

Reverse the roles of $x$ and $y$ to get $\boxed{y = f^{-1}(x) = \frac{1}{2}e^{x/(1-x)}}$

(b) Consider the function $g(x) = 2x - \cos x$.

i. Explain why $g$ is invertible, based on its derivative.

ii. Find an equation of the line that is tangent to the curve $y = g^{-1}(x)$ at the point $(4\pi - 1, 2\pi)$.
*Hint:* Do not attempt to identify the function $g^{-1}(x)$.

**Solution:**

i. $g'(x) = 2 + \sin x$, which is positive for all real numbers $x$ since $-1 \le \sin x \le 1$.

Therefore, $g(x)$ is a monotone increasing function, which implies that it is invertible.

ii. The slope of the line that is tangent to the curve $y = g^{-1}(x)$ at the point $(4\pi - 1, 2\pi)$ is $(g^{-1})'(4\pi - 1)$.

Since $(g^{-1})'(x) = \dfrac{1}{g'(g^{-1}(x))}$, we know that $(g^{-1})'(4\pi - 1) = \dfrac{1}{g'(g^{-1}(4\pi - 1))}$.

Since the curve $y = g^{-1}(x)$ passes through the point $(4\pi - 1, 2\pi)$, we know that $g^{-1}(4\pi - 1) = 2\pi$.

It follows that $(g^{-1})'(4\pi - 1) = \dfrac{1}{g'(2\pi)}$.

The expression for $g'(x)$ from part (i) implies that $g'(2\pi) = 2 + \sin(2\pi) = 2$. Therefore,

$(g^{-1})'(4\pi - 1) = \dfrac{1}{g'(2\pi)} = \dfrac{1}{2}$.

Since the tangent line passes through the point $(4\pi - 1, 2\pi)$ its equation is

$$\boxed{y - 2\pi = \frac{1}{2}(x - (4\pi - 1))}$$

2. (25 pts) Parts (a) and (b) are unrelated.

(a)

(b) Consider the function $p(t) = p_0 e^{kt}$, which represents an exponential growth model for a population, where the constant $p_0$ represents the initial population size and the constant $k$ represents the population's relative growth rate. Suppose $p(10) = 2$ and $p(50) = 6$.

i. Find the value of $k$.
ii. Find the value of $p_0$.

**Solution:**

The two given data points lead to the following system of two equations and two unknowns:

$$(t; p) = (10;$$

3. (26 pts) Evaluate the following derivatives using properties of logarithms and/or logarithmic differentiation. Do **not** fully simplify your answers, although they must be expressed as functions of $x$.

(a) $$\frac{d}{dx}\left[\ln\left(\frac{(10 - \cos^2 x)\sqrt{x^4+6}}{e^{x\sin x}}\right)\right]$$

**Solution:**

$$\begin{aligned}
\frac{d}{dx}\left[\ln\left(\frac{(10 - \cos^2 x)\sqrt{x^4+6}}{e^{x\sin x}}\right)\right] &= \frac{d}{dx}\left[\ln\left(10 - \cos^2 x\right) + \ln\left[(x^4+6)^{1/2}\right] - \ln\left[e^{x\sin x}\right]\right] \\
&= \frac{d}{dx}\left[\ln\left(10 - \cos^2 x\right) + \frac{1}{2}\ln(x^4+6) - x\sin x\right] \\
&= \frac{(-2\cos x)(-\sin x)}{10 - \cos^2 x} + \frac{1}{2}\left(\frac{4x^3}{x^4+6}\right) - (x\cos x + \sin x) \\
&= \boxed{\frac{2\cos x\sin x}{10 - \cos^2 x} + \frac{2x^3}{x^4+6} - x\cos x - \sin x}
\end{aligned}$$

(b) $\dfrac{d}{dx}\left[\left(e^x + e^{-x}\right)^x\right]$

**Solution:**

Let $y = (e^x + e^{-x})^x$.

$$\begin{aligned} \ln y &= \ln\left[\left(e^x + e^{-x}\right)^x\right] \\ &= x\ln\left(e^x + e^{-x}\right) \end{aligned}$$

$$\begin{aligned} \frac{d}{dx}[\ln y] &= \frac{d}{dx}\left[x\ln\left(e^x + e^{-x}\right)\right] \\ \frac{1}{y}\cdot\frac{dy}{dx} &= x\cdot\frac{e^x - e^{-x}}{e^x + e^{-x}} + \ln\left(e^x + e^{-x}\right) \\ \frac{dy}{dx} &= y\left[x\cdot\frac{e^x - e^{-x}}{e^x + e^{-x}} + \ln\left(e^x + e^{-x}\right)\right] \\ \frac{dy}{dx} &= \boxed{\left(e^x + e^{-x}\right)^x\left[x\cdot\frac{e^x - e^{-x}}{e^x + e^{-x}} + \ln\left(e^x + e^{-x}\right)\right]} \end{aligned}$$

4. (26 pts) Evaluate the following integrals.

(a) $\displaystyle\int_1^2 \frac{2^x}{9 - 2^x}\,dx$

**Solution:**

Let $u = 9 - 2^x$, which implies that $du = -2^x \ln 2\,dx$.

$x = 1 \;\Rightarrow\; u = 9 - 2^1 = 7$

$x = 2 \;\Rightarrow\; u = 9 - 2^2 = 5$

$$\int_1^2 \frac{2^x}{9 - 2^x}\,dx = -\frac{1}{\ln 2}\int_7^5 \frac{du}{u} = \frac{1}{\ln 2}\int_5^7 \frac{du}{u} = \boxed{\frac{\ln 7 - \ln 5}{\ln 2}}$$

(b) $\displaystyle\int \frac{x}{x - 1}\,dx$

**Solution:**

Let $u = x - 1$, which implies that $du = dx$ and $x = u + 1$.

$$\int \frac{x}{x - 1}\,dx = \int \frac{u + 1}{u}\,du = \int du + \int \frac{du}{u} = u + \ln|u| + C = \boxed{x - 1 + \ln|x - 1| + C}$$

Your Initials ________

ADDITIONAL BLANK SPACE

If you write a solution here, please clearly indicate the problem number.