1. (30 pts) Determine $\dfrac{dy}{dx}$ for each of the following.

   (a) $y = \sin^4(x^3)$

   (b) $x^2 + xy + y^3 = 4$

   (c) $y = \dfrac{2x^2+1}{x\cos x}$ After fully differentiating, do not algebraically simplify your answer any further.

Solution:

(a)

$$\frac{d}{dx}\ \sin^4(x^3) = 4\sin^3(x^3)\frac{d}{dx}\ \sin(x^3) = 4\sin^3(x^3)\cos(x^3)\frac{d}{dx}\ x^3$$

$$= 4\sin^3(x^3)\cos(x^3)(3x^2) = \boxed{12x^2\sin^3(x^3)\cos(x^3)}$$

(b)

$$\frac{d}{dx}\ x^2 + xy + y^3 = \frac{d}{dx}[4]$$

$$2x + xy' + y + 3y^2y' = 0$$

$$y'(x + 3y^2) = \quad (2x + y)$$

$$y' = \boxed{\ \frac{2x+y}{x+3y^2}}$$

(c)

$$\frac{d}{dx}\ \frac{2x^2+1}{x\cos x} = \frac{x\cos x\ \ \frac{d}{dx}[2x^2+1] \quad (2x^2+1)\ \ \frac{d}{dx}[x\cos x]}{(x\cos x)^2}$$

$$= \frac{x\cos x\ \ (4x) \quad (2x^2+1)\ \ x\ \ \frac{d}{dx}[\cos x] + \cos x\ \ \frac{d}{dx}[x]}{(x\cos x)^2}$$

$$= \boxed{\frac{x\cos x\ \ (4x) \quad (2x^2+1)(\ \ x\sin x + \cos x)}{(x\cos x)^2}}$$

2. (25 pts) Parts (a) and (b) are unrelated.

   (a) The position function of Particle P is given by $s(t) = 2{=}t + t{=}2$, where s is in meters, t is in seconds, and $t \quad 1$.

      i. Find the particle's velocity function $v(t)$. Include the correct unit of measurement.

3. (23 pts) Parts (a) and (b) are unrelated.

(a) Find the equations of the tangent and normal lines to the curve $y = x^{3/2} - x^{1/2}$ at $x = 4$.

(b) Find all values of x on the interval $[0, \pi]$ for which the curve $y = \sin^2 x - \sin x$ has a horizontal tangent line.

Solution:

(a) $y'(x) = \frac{3}{2}x^{1/2} - \frac{1}{2}x^{-1/2} = x^{-1/2}\left(\frac{3}{2}x - \frac{1}{2}\right) = \frac{x^{-1/2}}{2}(3x - 1) = \frac{3x - 1}{2\sqrt{x}}$

$y'(4) = \frac{11}{4}$

$y(4) = 4^{3/2} - 4^{1/2} = 8 - 2 = 6$

Tangent line: $\boxed{y - 6 = \frac{11}{4}(x - 4)}$

Normal line: $\boxed{y - 6 = -\frac{4}{11}(x - 4)}$

(b) $y'(x) = 2\sin x\cos x - \cos x = \cos x(2\sin x - 1) = 0$

$\cos x = 0 \Rightarrow x = \frac{\pi}{2}$

$2\sin x - 1 = 0 \Rightarrow \sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$

Therefore, $\boxed{x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}}$

4. (22 pts) Parts (a) and (b) are unrelated.

(a) Determine $f'(x)$ for the function $f(x) = \dfrac{1}{x+1}$ by using the **definition of derivative**.

You must obtain f' by evaluating the appropriate **limit** to earn credit.

(b) Find the values of b and c for which the following function $g(x)$ is differentiable at $x = 2$.

$$g(x) = \begin{cases} \dfrac{3}{8}x^3 & ; \quad x < 2 \\ -x^2 + bx + c & ; \quad x \geq 2 \end{cases}$$

You do **not** have to explicitly state the one-sided limits that are being evaluated.

Solution:

(a)

$$f'(x) = \lim_{h \to 0} \frac{\dfrac{1}{(x+h)+1} - \dfrac{1}{x+1}}{h} = \lim_{h \to 0} \frac{1}{h}\left(\frac{1}{x+h+1} - \frac{1}{x+1}\right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{(x+1) - (x+h+1)}{(x+h+1)(x+1)}\right) = \lim_{h \to 0} \frac{1}{h}\left(\frac{-h}{(x+h+1)(x+1)}\right)$$

$$= \lim_{h \to 0} \frac{-1}{(x+h+1)(x+1)} = \boxed{-\frac{1}{(x+1)^2}}$$

(b)

$$g'(x) = \begin{cases} \dfrac{9}{8}x^2 & ; \quad x < 2 \\ -2x + b & ; \quad x > 2 \end{cases}$$

In order for g to be differentiable at $x = 2$, we must have $\lim_{x \to 2^-} g'(x) = \lim_{x \to 2^+} g'(x)$, which leads to the following:

$$\frac{9}{8}\left(2^2\right) = (-2)(2) + b$$

$$\frac{9}{2} = -4 + b$$

$$b = \boxed{\frac{17}{2}}$$

In order for g to be differentiable at $x = 2$, g must also be continuous at $x = 2$.

$\lim\limits_{x \to 2} g(x) = \lim\limits_{x \to 2^+} g(x) = g(2)$ leads to the following:

$$\frac{3}{8}\ 2^3 = (2^2) + \frac{17}{2}(2) + c$$

$$3 = 4 + 17 + c$$

$$c = \boxed{10}$$